Appendix A

Set of algebraic equations F_1 that the pair of eigenvelocity vectors v and u and its eigenvalue β satisfy

Consider two vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ that relate to each other in general as

$$\boldsymbol{b}\boldsymbol{v}^i = \boldsymbol{a}^i_k \boldsymbol{u}^k \tag{A1}$$

$$\boldsymbol{b}'\boldsymbol{u}^i = \boldsymbol{a}'^i_k \boldsymbol{v}^k \tag{A2}$$

where $\boldsymbol{a}$ and $\boldsymbol{a}'$ are tensors of type (**1, 1**) and $\boldsymbol{b}$ and $\boldsymbol{b}'$ are two scalar variables. Rewriting (**A1**) and (**A2**) with $\boldsymbol{v}$ and $\boldsymbol{u}$ in their coordinate form we get

$$\boldsymbol{b}d\boldsymbol{x}^i/d\boldsymbol{s} = \boldsymbol{a}^i_k\, d'\boldsymbol{x}^k/d\boldsymbol{s}' \tag{A3}$$

$$\boldsymbol{b}'d'\boldsymbol{x}^i/d\boldsymbol{s}' = \boldsymbol{a}'^i_k\, d\boldsymbol{x}^k/d\boldsymbol{s} \tag{A4}$$

where $\boldsymbol{x}^0$ is the time coordinate and $\boldsymbol{s}$ and $\boldsymbol{s}'$ are path parameters. Let $\boldsymbol{b}$, $\boldsymbol{b}'$ and the two path parameters be adjusted so that

$$\boldsymbol{b}(d\boldsymbol{s}'/d\boldsymbol{s}) = \boldsymbol{b}'(d\boldsymbol{s}/d\boldsymbol{s}') = \boldsymbol{\beta} \tag{A5}$$

where $\boldsymbol{\beta}$ is a scalar variable. Then (**A1**) and (**A2**) become

$$\boldsymbol{\beta}\boldsymbol{v}^i = \boldsymbol{a}^i_k \boldsymbol{u}^k \tag{A6}$$

$$\boldsymbol{\beta}\boldsymbol{u}^i = \boldsymbol{a}'^i_k \boldsymbol{v}^k \tag{A7}$$

where

$$\boldsymbol{v}^i = d\boldsymbol{x}^i/d\boldsymbol{s} \tag{A8}$$

$$\boldsymbol{u}^i = d'\boldsymbol{x}^i/d\boldsymbol{s} \tag{A9}$$

The path parameter $\boldsymbol{s}$ is the ET-equivalent of proper time in Relativity.

Equations (**A6**) and (**A7**) can be re-written as follows.

$$\boldsymbol{\beta}\boldsymbol{g}_{ij}\boldsymbol{v}^j = (\boldsymbol{g}_{ij} + \boldsymbol{h}_{ij})\boldsymbol{u}^j \tag{A10}$$

$$\boldsymbol{\beta}\boldsymbol{g}'_{ij}\boldsymbol{u}^j = (\boldsymbol{g}'_{ij} + \boldsymbol{h}'_{ij})\boldsymbol{v}^j \tag{A11}$$

where $\boldsymbol{g}$ and $\boldsymbol{g}'$ are symmetric and $\boldsymbol{h}$ and $\boldsymbol{h}'$ are antisymmetric. The condition that $\boldsymbol{v}$ and $\boldsymbol{u}$ are eigenvectors is $\boldsymbol{a}'$ in (**A7**) is the transpose of $\boldsymbol{a}$ in (**A6**) (ref.1). This condition is satisfied if

$$\boldsymbol{g}' = \boldsymbol{g} \tag{A12}$$

$$\boldsymbol{h}' = -\boldsymbol{h} \tag{A13}$$

Re-writing (**A10**) and (**A11**) to include (**A12**) and (**A13**) we have

$$\boldsymbol{\beta}\boldsymbol{g}_{ij}\boldsymbol{v}^j = (\boldsymbol{g}_{ij} + \boldsymbol{h}_{ij})\boldsymbol{u}^j \tag{A14}$$

$$\boldsymbol{\beta}\boldsymbol{g}_{ij}\boldsymbol{u}^j = (\boldsymbol{g}_{ij} - \boldsymbol{h}_{ij})\boldsymbol{v}^j \tag{A15}$$

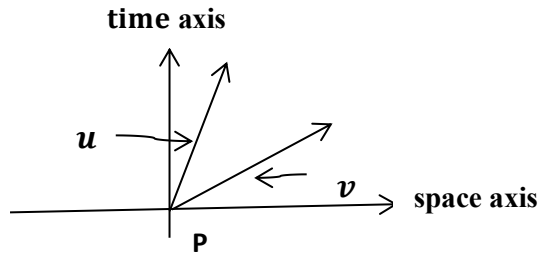

Figure: A1 Time and space axes coplanar with ($\boldsymbol{v}$, $\boldsymbol{u}$)

In Minkowski spacetime let the coordinate axes at the point of intersection of $\boldsymbol{v}$ and $\boldsymbol{u}$ be so configured that $\boldsymbol{v}$ and $\boldsymbol{u}$ are coplanar with the time-axis and a space-axis, as shown in Fig. A1.

In this case, (**A14**) and (**A15**) produce the following.

$$\sqrt{1-s^2}\begin{bmatrix} v^0 \\ v^1 \end{bmatrix} = \begin{bmatrix} 1 & s \\ s & 1 \end{bmatrix}\begin{bmatrix} u^0 \\ u^1 \end{bmatrix} \tag{A16}$$

$$\sqrt{1-s^2}\begin{bmatrix} u^0 \\ u^1 \end{bmatrix} = \begin{bmatrix} 1 & -s \\ -s & 1 \end{bmatrix}\begin{bmatrix} v^0 \\ v^1 \end{bmatrix} \tag{A17}$$

$$s = h_{01} \tag{A18}$$

According to these the two eigenvelocity vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ are Lorentzian boosts of each other with a boost speed of s.

Appendix B

The mutual transport equations of v and u

Let an arbitrary infinitesimal perturbation δx^i, $i = 0,\ldots n\text{-}1$, be applied to the point of intersection of the pair of eigenvelocity vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ so that $\boldsymbol{\beta}$ is kept invariant. Then according equations (**5**) in §**1.2** we have

$$\delta(g_{ij}v^i u^j) = \delta\beta = 0 \tag{B1}$$

$$\delta(h_{ij}v^i u^j) = \delta\beta(1-\beta^2) = 0 \tag{B2}$$

On expanding (**B1**) we get

$$\left[g_{kn}\frac{d}{ds}\left\{\frac{d'x^n}{ds}\right\} + g_{kn}\frac{d'}{ds}\left\{\frac{dx^n}{ds}\right\} + 2g_{ko}\{ij,o\}^s\frac{d'x^i}{ds}\frac{dx^j}{ds}\right]\delta x^k = \beta_g \tag{B3}$$

where

$$\{ij,k\}^s = \frac{1}{2}g^{lk}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^l}\right) \tag{B4}$$

$$\beta_g = \frac{d}{ds}\left\{g_{ij}\delta x^i\frac{d'x^j}{ds}\right\} + \frac{d'}{ds}\left\{g_{ij}\frac{dx^i}{ds}\delta x^j\right\} \tag{B5}$$

Since δx^i is arbitrary, (**B3**) effectively represents n equations, which respectively correspond to the n elements of δx.

Let δx^l be an element of δx with the remaining elements of δx set to zero. Let the scalars within the curly brackets in (**B5**) be denoted by c'^l and c^l as follows.

$$g_{lj}\delta x^l \frac{d'x^j}{ds} = c'^l \quad \textbf{(B5a)}$$

$$g_{lj}\delta x^l \frac{dx^j}{ds} = c^l \quad \textbf{(B5b)}$$

The summation on l is suspended in (**B5a**) and (**B5b**). Eliminating δx^l from (**B5a**) and (**B5b**) we get

$$(g_{lj}/c'^l)\frac{d'x^j}{ds} = (g_{lj}/c^l)\frac{dx^j}{ds} \quad \textbf{(B5c)}$$

In the n independent equations that correspond to l =0,…n-1, which (B5c) represents, the ratios between c'^l and c^l can be maintained constant using the set of n arbitrary elements of g. Then because x^l is arbitrary, c'^l and c^l can be individually maintained constant. Therefore

$$\beta_g = 0 \quad \textbf{(B6)}$$

Then we have

$$\left[g_{kn}\frac{d}{ds}\left\{\frac{d'x^n}{ds}\right\} + g_{kn}\frac{d'}{ds}\left\{\frac{dx^n}{ds}\right\} + 2g_{ko}\{ij,o\}^s\frac{d'x^i}{ds}\frac{dx^j}{ds}\right] = 0 \quad \textbf{(B7)}$$

On expanding (**B2**) in the same way that (**B1**) was expanded, we get

$$\left[h_{kn}\frac{d}{ds}\left\{\frac{d'x^n}{ds}\right\} - h_{kn}\frac{d'}{ds}\left\{\frac{dx^n}{ds}\right\} + 2h_{ko}\{ij,o\}^a\frac{d'x^i}{ds}\frac{dx^j}{ds}\right]\delta x^k = \beta_h \quad \textbf{(B8)}$$

where

$$\{ij,k\}^a = \frac{1}{2}h^{lk}\left(\frac{\partial h_{il}}{\partial x^j} + \frac{\partial h_{lj}}{\partial x^i} + \frac{\partial h_{ji}}{\partial x^l}\right) \quad \textbf{(B9)}$$

$$\beta_h = \frac{d}{ds}\left\{h_{ij}\delta x^i\frac{d'x^j}{ds}\right\} + \frac{d'}{ds}\left\{h_{ij}\frac{dx^i}{ds}\delta x^j\right\} \quad \textbf{(B10)}$$

The reasoning that was used to obtain (**B6**) is applicable in this case also as the 1-form that h contains consists of n arbitrary elements. Thus, we get

$$\beta_h = 0 \quad \textbf{(B11)}$$

$$\left[h_{kn}\frac{d}{ds}\left\{\frac{d'x^n}{ds}\right\} - h_{kn}\frac{d'}{ds}\left\{\frac{dx^n}{ds}\right\} + 2h_{ko}\{ij,o\}^a\frac{d'x^i}{ds}\frac{dx^j}{ds}\right] = 0 \quad \textbf{(B12)}$$

Equations (**B7**) and (**B12**) can be re-written as

$$\frac{d}{ds}\left\{\frac{d'x^k}{ds}\right\} + \frac{d'}{ds}\left\{\frac{dx^k}{ds}\right\} + 2\{ij,k\}^s\frac{d'x^i}{ds}\frac{dx^j}{ds} = 0 \quad \textbf{(B13)}$$

$$\frac{d}{ds}\left\{\frac{d'x^k}{ds}\right\} - \frac{d'}{ds}\left\{\frac{dx^k}{ds}\right\} + 2\{ij,k\}^a\frac{d'x^i}{ds}\frac{dx^j}{ds} = 0 \quad \textbf{(B14)}$$

Adding (**B13**) and (**B14**) and subtracting (**B14**) from (**B13**) we get the following equations that determine the mutual transport of v and u.

$$\frac{d}{ds}\left\{\frac{d'x^k}{ds}\right\} + \{ij,k\}\frac{d'x^i}{ds}\frac{dx^j}{ds} = 0 \quad \textbf{(B15)}$$

$$\frac{d'}{ds}\left\{\frac{dx^k}{ds}\right\} + \{ij,k\}\frac{dx^i}{ds}\frac{d'x^j}{ds} = 0 \quad \textbf{(B16)}$$

where

$$\{ij,k\} = \{ij,k\}^s + \{ij,k\}^a \quad \textbf{(B17)}$$

Appendix C

A simple generalisation of Maxwell's electromagnetic equations

In ET, the antisymmetric tensor h is the sum of a 2-form and the exterior derivative of a 1-form p. This h can be used to generalise the following Maxwell's electromagnetic equations

$$F_{ij=}\ \partial_j p_i - \partial_i p_j \quad \textbf{(C1)}$$

$$\partial_j\{\sqrt{-g}g^{im}g^{jn}F_{mn}\} = \sqrt{-g}J^i \quad \textbf{(C2)}$$

where J is the charge-current density vector. If the 2-form component of h, say h, is assumed to be related to J as

$$\partial_j\{\sqrt{-g}g^{im}g^{jn}\underline{h}_{mn}\} = -\sqrt{-g}J^i \quad \textbf{(C3)}$$

then in **ET** the above Maxwell's equations become generalised as

$$\partial_j\{\sqrt{-g}g^{im}g^{jn}h_{mn}\} = 0 \quad \textbf{(C4)}$$

Appendix D

A simple insight into Planck-Einstein Relation

In 2-dimensional Minkowski spacetime, equation

$$\left(\overline{g}_{ij} \pm \overline{h}_{ij}\right)\underline{w}^i = 0, \quad \det(\overline{g} \pm \overline{h}) = 0 \quad \textbf{(22)}$$

reduces to just the following.

$$dt/\,d\mathbf{s} = dx/d\mathbf{s} \quad \textbf{(D1)}$$

where dt and dx are the time and space increments of the photon travel and $d\mathbf{s}$ is path parameter increment yet to be established owing to the null character of $\underline{w}$. Now dt/ds is the dimensionless photon energy E'. Therefore, (D1) becomes

$$E' = dx/ds \quad \textbf{(D2)}$$

Since this energy E' is dimensionless it has to be multiplied by a fundamental energy constant to obtain E in the usual units of energy. This constant is Planck energy $(hc/G)^{1/2}c^2$ where $\hbar$ is Planck's reduced constant, c is speed of light and G is gravitation constant. Then (**D2**) becomes

$$E = (\hbar c/G)^{1/2}c^2(dx/ds) \quad \textbf{(D3)}$$

Now (22) is the result of unification of translational and rotational motions of light. This unification in this 2-dimensional case connects dx with an incremental hyperbolic angle dξ using Planck distance as follows.

$$dx = (Planck\ distance)d\xi \quad \textbf{(D4)}$$

Since Planck distance is $(\hbar G/c^3)^{1/2}$, (*D3*) becomes

$$E = (\hbar c)(d\xi/ds) \quad \textbf{(D5)}$$

If the normal angular velocity is the same as $d\xi/(ds/c)$ then that would define the path parameter s and (**D5**) becomes the same as Planck-Einstein relation.

$$E = h\nu \quad \textbf{(D6)}$$

where ν is normal frequency.

Appendix E

Solution of (25m), 26m) and (24nm) in §2.4, for static spherically symmetric space

Equations (**25m**), (**26m**) and their auxiliary equations are as follows.

$$\underline{R}_{ij} + \underline{C}^s_{ij} = \underline{b}\underline{g}_{ij} \quad \textbf{(25m)}$$

$$\underline{C}^a_{ij} = \pm\underline{b}\underline{h}_{ij} \quad \textbf{(26m)}$$

$$\det\left(\underline{g} \pm \underline{h}\right) = 0 \quad \textbf{(24nm)}$$

where $\underline{R}_{ij} = \{ik,k\}^s_{,j} - \{ij,k\}^s_{,k} - \{mk,k\}^s\{ij,m\}^s$ **(27)**

$$+\{mj,k\}^s\{ik,m\}^s$$

$$\underline{C}^s_{ij} = \{ik,k\}^a_{;j} + \{mj,k\}^a\{ik,m\}^a \quad \textbf{(28)}$$

$$\underline{C}^a_{ij} = -\{mk,k\}^a\{ij,m\}^a - \{ij,k\}^a_{,k} \quad \textbf{(29)}$$

$$\{ij,k\}^s = ½\underline{g}^{lk}\left(\partial_j\underline{g}_{il} + \partial_i\underline{g}_{lj} - \partial_l\underline{g}_{ji}\right) \quad \textbf{(30)}$$

$$\{ij,k\}^a = ½\underline{h}^{lk}\left(\partial_j\underline{h}_{il} + \partial_i\underline{h}_{lj} + \partial_l\underline{h}_{ji}\right) \quad \textbf{(31)}$$

For static spherically symmetric conditions $\underline{g}$ and $\underline{h}$ in spherical polar coordinates (t, r, θ, ϕ) have the following forms.

$$\underline{g}_{ij} = diag\{e^\nu, -e^\lambda, -r^2, -r^2\sin^2\theta\} \quad \textbf{(E1)}$$

$$\underline{h}_{ij} = \{\ \underline{h}_{01} = -\ \underline{h}_{10} = -\ e^\alpha,\ \underline{h}_{23} = -\ \underline{h}_{32} = \sin\theta e^\rho\} \quad \textbf{(E2)}$$

Parameters ν, λ, α and ρ are functions of r only and all other elements of $\underline{h}$ are zero. The elements of $\{ij,k\}^s$ that correspond to $\underline{g}$ in (**E1**) are well established in the literature and they can be found on page 84 of ref. 2. The non-zero elements of $\{ij,k\}^a$ that correspond to $\underline{h}$ in (**E2**) is as follows.

$$\{12,2\}^a = -\rho'/2 \quad \textbf{(E3)}$$

$$\{13,3\}^a = -\ \rho'/2 \quad \textbf{(E4)}$$

$$\{21,2\}^a = +\rho'/2 \quad \textbf{(E5)}$$

$$\{23,0\}^a = +\sin\theta e^{\rho-\alpha}\rho'/2 \quad \textbf{(E6)}$$

$$\{31,3\}^a = +\rho'/2 \quad \textbf{(E7)}$$

$$\{32,0\}^a = -\sin\theta e^{\rho-\alpha}\rho'/2 \quad \textbf{(E8)}$$

The accent,' on a symbol denotes differentiation with respect to r. On substituting these elements of $\{ij,k\}^a$ in the expression for $\underline{C}^a_{ij}$ in (29), we get

$$\underline{C}^a_{ij} = 0 \quad \textbf{(E9)}$$

Hence, it follows that the parameter $\underline{b}$ in (**25m**) and (**26m**) is zero. On substituting the above elements of $\{ij,k\}^a$ in the expression for $\underline{C}^s_{ij}$ in (**28**), we get the following.

$$\underline{C}^s_{00} = e^{\nu-\lambda}\nu'(\rho')/2 \quad \textbf{(E10)}$$

$$\underline{C}^s_{11} = -(\rho'') + \lambda'(\rho')/2 - (\rho')^2/2 \quad \textbf{(E11)}$$

$$\underline{C}^s_{22} = -re^{-\lambda}(\rho') \quad \textbf{(E12)}$$

$$\underline{C}^s_{33} = -r\sin^2\theta e^{-\lambda}(\rho') \quad \textbf{(E13)}$$

Elements $\{ij,k\}^s$ of the tensor $\underline{R}_{ij}$ in (**27**) can be found on page **85** of ref. 2 where R has been denoted as G. On substituting in (**25m**) these elements $\{ij,k\}^s$ and the above elements of, $\underline{C}^s_{ij}$ we get

$$e^{\nu-\lambda}\left(-\frac{1}{2}\nu'' + \frac{1}{4}\lambda'\nu' - \frac{1}{4}\nu'^2 - \frac{\nu'}{r}\right) + \frac{1}{2}e^{\nu-\lambda}\nu'\rho' = 0 \quad \textbf{(E14)}$$

$$\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 - \frac{\lambda'}{r} - (\rho'') + \frac{1}{2}\lambda'(\rho') - \frac{1}{2}(\rho')^2 = 0 \quad \textbf{(E15)}$$

$$e^{-\lambda}\left(1 + \frac{1}{2}r(\nu' - \lambda')\right) - 1 - re^{-\lambda}(\rho') = 0 \quad \textbf{(E16)}$$

The parameters ν, λ and ρ that satisfy equations (**E14**) to (**E16**) are as follows.

$$\rho' = 2/r \tag{E17}$$

$$e^{\nu} = -(1 - r/(2M)) \tag{E18}$$

$$e^{\lambda} = -1/(1 - r/(2M)) \tag{E19}$$

M in (**E18**) and (**E19**) is a constant of integration. Applying the condition (**24nm**) to the parameters in (**E17**) to (**E19**), $\underline{g}$ and $\underline{h}$ in (**E1**) and (**E2**), become

$$\underline{g}_{ij} = diag\{-\left(1 - \frac{r}{2M}\right), 1/\left(1 - \frac{r}{2M}\right), -r^2, -r^2 sin^2\theta\} \tag{E20}$$

$$\underline{h}_{ij} = \{\, \underline{h}_{01} = -\underline{h}_{10} = -1, \; \underline{h}_{23} = -\underline{h}_{32} = kr^2 sin\theta\} \tag{E21}$$

where k is a constant which may be set to unity.

Appendix F

Gravitational redshift due to spacetime curvature of the innermost manifold

The metric tensor field of the innermost manifold, obtained in Appendix E, for static spherically symmetric space is the following.

$$\underline{g}_{ij} = diag\{-\left(1 - \frac{r}{2M}\right), 1/\left(1 - \frac{r}{2M}\right), -r^2, -r^2 sin^2\theta\} \tag{E20}$$

Let a photon be emitted at a distance r from the origin O of the system of spherical polar coordinates (t, r, θ, ϕ), and let its frequency at this point of emission be ν_e. On reaching O let the photon frequency become ν_r. These two frequencies relate to each other as (**ref.4**)

$$\frac{\nu_e}{\nu_r} = (1 - r/(2M))^{-1/2} \tag{F1}$$

This frequency ratio in terms of a recessional speed s, is given by

$$\frac{\nu_e}{\nu_r} = \sqrt{\frac{1+s}{1-s}} \tag{F2}$$

Combining (**F1**) and (**F2**), we get

$$s = \frac{r/(2M)}{2 - r/(2M)} \tag{F3}$$

For comparing with Hubble's law, let (**F3**) be re-written as $s = \underline{H}r$, where

$$\underline{H} = \frac{1}{4M - r} \tag{F4}$$

According to (**F4**) the maximum value of $\underline{H}$ is **1/(2*M*)** that occurs at $r = 2M$. Now M has been estimated in §**6.4** as **7.47 x 10^{56} g** which works out to **1.799x10^{07} kpc**. Hence the maximum value of H **(=1/(2*M*))** is **8.34 kms^{-1}Mpc^{-1}**. This value of $\underline{H}$ is considerably less than the present value of the Hubble's constant, **73.8 kms^{-1} Mpc^{-1}**. Therefore, the gravitational curvature of the innermost manifold has only a small effect on the observed Hubble expansion of the physical universe.

References

1. George Arfken (1985) Geometrical Methods for Physicists. Academic Press Inc 235.
2. Eddington AS (1975) The Mathematical Theory of Relativity. Chelsea Publishing Company 84-85.
3. Corbelli E., Salucci P (2000) The Extended Rotation Curve and the Dark Matter Halo of M33. Monthly Notices of the Royal Astronomical Society 311: 441- 447
4. Schutz BF (1985) A First Course in General Relativity. Cambridge University Press 255.