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On Generalized Einstein Field Equation Geodesic Equation Conservation Laws and Torsion Tensor in a Non-Symmetric Geometry

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Abstract

In four dimensions, the Minkowski metric $\eta_{\mu\nu}=diag(+1,-1,-1,-1)$ leads to the 16-dimensional Clifford algebra C(1,3), Dirac's equation [1] is using four of these 16 matrices that form a basis of this algebra, a new operator is defined using all of these matrices and also generalized for a curved space. This new multilevel operator generalizes the Dirac's equation, the value of the generalized Dirac's operator is calculated in the Schwarzschild's metric. The torsion tensor is calculated taking into account the non-symmetric part of the metric tensor in the vanishing of its covariant derivative and applied to Kerr's metric generalizing the Clifford algebra. Geodesic equation, conservation laws, torsion tensor and Einstein field equation are obtained in a non-symmetric geometry.

Keywords: *Gravitomagnetic Tensor; Gravitational Magnetic Field; Energymomentum1-Form; Clifford Algebra, Dirac Equation; Dirac Operator; Gravity and Quantum Mechanics Unification; Multilevel Operator; Schwarzschild's Metric; Torsion Tensor; Rearranged Kerr's Metric; Generalized Clifford Algebra; Generalized Einstein Field Equation; Generalized Geodesic Equation; Conservation Laws; Non-Symmetric Geometry*

Introduction

Dirac's equation is the relativistic wave equation derived by physicist Paul Dirac in 1928. The wave functions in the Dirac theory are vectors of four complex components (known as bispinors), two of which resemble the Pauli wavefunction in the non-relativistic limit, in contrast to the Schrodinger equation which described wave functions of only one complex component.

Dirac's operator is just the tip of the iceberg, the tip of a generalized operator that is obtained by operating on all members of the Clifford algebra basis and not just on four of them.

The Schwarzschild's metric is named in honour of Karl Schwarzschild, who found the exact solution in 1915 and published it in January 1916, a little more than a month after the publication of Einstein's theory of general relativity. It was the first exact solution of the Einstein field equations other than the trivial at space solution. Schwarzschild died shortly after his paper was published, as a result of a disease he developed while serving in the German army during World War I. Johannes Droste in 1916 independently produced the same solution as Schwarzschild.

Schwarzschild's metric is an exact solution to the Einstein's field equations that describes the gravitational field outside a spherical mass, on the assumption that the electric charge of the mass, angular momentum of the mass, and universal cosmological constant is all zero.

The new generalized Dirac's operator, the multilevel operator, is calculated in the Schwarzschild's metric, torsion tensor and new gravitomagnetic tensor appear in level 2, curvature tensor appears in levels 3 and 4.

The Kerr's metric is a generalization to a rotating body of the Schwarzschild's metric. The Einstein field equation relates the geometry of spacetime to the distribution of matter within it. The equations were published by Einstein in 1915 in the form of a tensor equation which related the local spacetime curvature with the local energy, momentum and stress within that spacetime expressed by the stress-energy tensor.

Multilevel operator D^{mul}

We are using Pauli matrices σ, electromagnetic four-potential A_μ and charge e with $\hbar=c=1$

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$
$$\alpha=\begin{pmatrix}0&\sigma\\\sigma&0\end{pmatrix}\quad \Sigma=\begin{pmatrix}\sigma&0\\0&\sigma\end{pmatrix} \qquad 1$$

In four dimensions, Minkowski's metric $\eta_{\mu\nu}=diag(+1,-1,-1,-1)$ leads to the Clifford algebra C(1,3)[2], $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}\times I_{4\times4}$, Dirac matrices $\gamma^0=\sigma_3\otimes I$, $\gamma^j=i\sigma_2\otimes\sigma_j$, j=1,2,3; $\gamma^p=-i\gamma^{14}=-i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\gamma^4=\gamma^0\gamma^1,\gamma^5=\gamma^0\gamma^2,\gamma^6=\gamma^0\gamma^3,\gamma^7=\gamma^1\gamma^2,\gamma^8=\gamma^1\gamma^3,\gamma^9=\gamma^2\gamma^3$$

$$\gamma^{10}=\gamma^0\gamma^1\gamma^2,\gamma^{11}=\gamma^0\gamma^1\gamma^3,\gamma^{12}=\gamma^0\gamma^2\gamma^3,\gamma^{13}=\gamma^1\gamma^2\gamma^3,\gamma^{14}=\gamma^0\gamma^1\gamma^2\gamma^3 \qquad 2$$

Multilevel operator D^n acts on level n, n is the number of matrices in the product of the algebra members, for example, D^3 acts on $\gamma^{10},\gamma^{11},\gamma^{12}$ and γ^{13}. Total multilevel operator $D^{mul} = D^0 + D^1 + D^2 + D^3 + D^4$, the action of Dmul on the spinor function vanishes $D^{mul}\,\Psi=0$ 3

$$D^0 = -m \quad (4)$$

$$D^1 = \gamma^\mu p_\mu - ie\gamma^\mu A_\mu \quad (5)$$

$$D^2 = -ie\gamma^\mu\gamma^\nu F_{\mu\nu} \text{ with } F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \quad (6)$$

$$D^2 = -ie\alpha E + e\Sigma H \quad (7)$$

$$D^3 = -ie\gamma^\mu\gamma^\nu\gamma^\delta F_{\mu\nu\delta} \text{ with } F_{\mu\nu\delta} = A_{\mu},_{\nu},_{\delta} - A_{\mu},_{\delta},_{\nu} = 0 \quad (8)$$

$$D^4 = -ie\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\lambda F_{\mu\nu\delta\lambda} \text{ with } F_{\mu\nu\delta\lambda} = A_{\mu},_{\nu},_{\delta},_{\lambda} - A_{\mu},_{\nu},_{\lambda},_{\delta} = 0 \quad (9)$$

Multilevel operator D^{mul} $(\eta_{\mu\nu}, A_\mu, e)$ can be generalized for a curved space with four-potential P, field charge q and covariant derivative[3] $(;_\mu)$ instead of derivative $(,_\mu)$ in the definition of p_μ

$$D^{mul}(g_{\mu\nu}, P_\mu, q)\Psi = 0 \quad (10)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbb{I} \quad (11)$$

$$D^0 = -m \quad (12)$$

$$D^1 = \gamma^\mu p_\mu - iq\gamma^\mu P_\mu \quad (13)$$

$$D^2 = -iq\gamma^\mu\gamma^\nu G_{\mu\nu} \text{ with } G_{\mu\nu} = P_{\mu;\nu} - P_{\nu;\mu} \quad (14)$$

$$G_{\mu\nu}(P) = P_{\mu;\nu} - P_{\nu;\mu} = P_{\mu,\nu} - P_{\nu,\mu} + P_\alpha T^\alpha_{\mu\nu} \quad (15)$$

$$G_{\mu\nu}(P) = F_{\mu\nu}(P) + P_\alpha T^\alpha_{\mu\nu} \quad (16)$$

$$D^2 = -iq\alpha E(P) + q\Sigma H(P) - iq\tfrac{1}{2}\gamma^\mu\gamma^\nu P_\alpha T^\alpha_{\mu\nu} \quad (17)$$

For gravity $G_{\mu\nu}(P)$ is the new gravitomagnetic tensor. $T^\alpha_{\mu\nu}$ is the torsion tensor [4].

$$D^3 = -iq\gamma^\mu\gamma^\nu\gamma^\delta G_{\mu\nu\delta} \text{ with } G_{\mu\nu\delta} = P_{\mu;\nu;\delta} - P_{\mu;\delta;\nu} \quad (18)$$

$G_{\mu\nu\delta}(P) = P_\alpha R^\alpha_{\mu\nu\delta}$ with $R^\alpha_{\mu\nu\delta}$ the Riemann-Christoffel tensor [5]. 19

$$D^3 = -iq\gamma^0\gamma^1\gamma^2 P_\alpha R^\alpha_{012} - iq\gamma^0\gamma^1\gamma^3 P_\alpha R^\alpha_{013} - iq\gamma^0\gamma^2\gamma^3 P_\alpha R^\alpha_{023}$$

$$-iq\gamma^1\gamma^2\gamma^3 P_\alpha R^\alpha_{123} \quad (20)$$

$$D^4 = -iq\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\lambda G_{\mu\nu\delta\lambda} \text{ with } G_{\mu\nu\delta\lambda} = P_{\mu;\nu;\delta;\lambda} - P_{\mu;\nu;\lambda;\delta} \quad (21)$$

$$G_{\mu\nu\delta\lambda}(P) = P_{\alpha;\nu} R^\alpha_{\mu\delta\lambda} + P_{\mu;\alpha} R^\alpha_{\nu\delta\lambda} \quad (22)$$

$$D^4 = -iq\gamma^0\gamma^1\gamma^2\gamma^3 P_{\alpha;1} R^\alpha_{023} - iq\gamma^0\gamma^1\gamma^2\gamma^3 P_{0;\alpha} R^\alpha_{123} \quad (23)$$

Gravitomagnetic tensor defined in Schwarzschild's Metric

We are using $x^0=t, x^1=r, x^2=\theta, x^3=\phi$ with $G = c = 1$, this metric is defined by [6]

$$ds^2 = (1-\tfrac{2M}{r})dt^2 - (1-\tfrac{2M}{r})^{-1}dr^2 - r^2d\theta^2 - r^2sin^2\theta d\phi^2 \quad (24)$$

$$g_{00} = (1-\tfrac{2M}{r}), g_{11} = -(1-\tfrac{2M}{r})^{-1}, g_{22} = -r^2, g_{33} = -r^2sin^2\theta \quad (25)$$

$$g^{00} = (1-\tfrac{2M}{r})^{-1}, g^{11} = -(1-\tfrac{2M}{r}), g^{22} = -r^{-2}, g^{33} = -r^{-2}sin^{-2}\theta \quad (26)$$

$$\Gamma^1_{00} = g_{00}Mr^{-2} \quad (27)$$

$$\Gamma^0_{01} = g_{00}^{-1}Mr^{-2} \quad (28)$$

$$\Gamma^1_{11} = -g_{00}^{-1}Mr^{-2} \quad (29)$$

$$\Gamma^2_{12} = \Gamma^3_{13} = r^{-1} \quad (30)$$

$$\Gamma^1_{22} = -g_{00}r \quad (31)$$

$$\Gamma^3_{23} = cot\theta \quad (32)$$

$$\Gamma^1_{33} = -g_{00}rsin^2\theta \quad (33)$$

$$\Gamma^2_{33} = -sin\theta cos\theta \quad (34)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbb{I} \quad (35)$$

$$\gamma^0 = g_{00}^{-1/2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (36)$$

$$\gamma^1 = -g_{00}^{1/2}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \quad (37)$$

$$\gamma^2 = -r^{-1}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \quad (38)$$

$$\gamma^3 = -r^{-1}sin^{-1}\theta\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \quad 39$$

$D^2 = im\alpha E(P) - m\Sigma H(P) + im\frac{1}{2}\gamma^\mu\gamma^\nu P_\alpha T^\alpha_{\mu\nu}$, from equations (16) and (17) 40

$$G_{12} = -H_3(P) = (-g_{00}^{1/2})(-r^{-1})(P_{1,2} - P_{2,1}) \quad 41$$

$$G_{13} = H_2(P) = (-g_{00}^{1/2})(-r^{-1}sin^{-1}\theta)(P_{1,3} - P_{3,1}) \quad 42$$

$$G_{23} = -H_1(P) = (-r^{-1})(-r^{-1}sin^{-1}\theta)(P_{2,3} - P_{3,2}) \quad 43$$

$$G_{03} = -E_3(P) = (g_{00}^{-1/2})(-r^{-1}sin^{-1}\theta)(P_{0,3} - P_{3,0}) \quad 44$$

$$G_{02} = -E_2(P) = (g_{00}^{-1/2})(-r^{-1})(P_{0,2} - P_{2,0}) \quad 45$$

$$G_{01} = -E_1(P) = (g_{00}^{-1/2})(-g_{00}^{1/2})(P_{0,1} - P_{1,0}) \quad 46$$

$$T^\alpha_{\mu\nu} = 0 \text{ and } R^\alpha_{012} = R^\alpha_{013} = R^\alpha_{023} = R^\alpha_{123} = 0 \quad 47$$

Energy-momentum form is a 1-form [7]

$$\mathbf{p} = E\mathbf{d}t - p_x\mathbf{d}x - p_y\mathbf{d}y - p_z\mathbf{d}z \quad 48$$

$\mathbf{dp}$ is a 2-form

$$\mathbf{G} = \mathbf{dp} = E_x\mathbf{d}t \wedge \mathbf{d}x + E_y\mathbf{d}t \wedge \mathbf{d}y + E_z\mathbf{d}t \wedge \mathbf{d}z - B_x\mathbf{d}y \wedge \mathbf{d}z - B_y\mathbf{d}z \wedge \mathbf{d}x - B_z\mathbf{d}x \wedge \mathbf{d}y \quad 49$$

$$G_{32} = p_{z,y} - p_{y,z} \quad 50$$

$$G_{13} = p_{x,z} - p_{z,x} \quad 51$$

$$G_{21} = p_{y,x} - p_{x,y} \quad 52$$

Comparing equations (41-43) and (50-52) we can infer

$$P_\alpha = p_\alpha \quad 53$$

D^0 is related to the scalar 0-form m, D^1 is related to the Energy-momentum 1-form, D^2 is related to the Electromagnetic 2-form, D^3 is related to $*\mathbf{J}$ 3-form [8]

$$\begin{pmatrix} *J_{123} \\ *J_{023} \\ *J_{013} \\ *J_{012} \end{pmatrix} = \begin{pmatrix} -\rho \\ j_1 \\ -j_2 \\ j_3 \end{pmatrix} \quad 54$$

D^4 is related to $\mathbf{L}$ 4-form [9]

$$\mathbf{L} = L_{0123}\mathbf{d}x^0 \wedge \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 \quad 55$$

$\gamma^p = -i\gamma^{14}$ is the proyector matrix, historically γ^5, but $\gamma^5 = \gamma^0\gamma^2$

$$\gamma^p = -i\gamma^0\gamma^1\gamma^2\gamma^3 = (g_{00}^{-1/2})(-g_{00}^{1/2})(-r^{-1})(-r^{-1}sin^{-1})\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \quad 56$$

Torsion Tensor in a Rearranged Kerr's Metric

We are using $x^0=t, x^1=r, x^2=\theta, x^3=\phi$ M is the black hole's mass and a is the angular momentum per unit mass with G = c = 1. The invariance of the length of vectors under parallel transport means that the connection is compatible with the metric, it is a metric connection, the requirement of the preservation of the length by parallel transport may be stated as [10].

$$g_{\mu\nu;\sigma} = 0 \quad 57$$

$$g_{\mu\nu;\sigma} = g_{\mu\nu,\sigma} - g_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} - g_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} \quad 58$$

$$0 = g_{\mu\nu,\sigma} - g_{\nu\alpha}\Gamma^\alpha_{\mu\sigma} - t_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} - g_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} \text{ , with } t_{\mu\nu} = g_{\mu\nu} - g_{\nu\mu} \quad 59$$

$$0 = g_{\mu\nu,\sigma} - g_{\nu\alpha}\Gamma^\alpha_{\mu\sigma} - t_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} - g_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} \quad 60$$

$$g_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} + g_{\nu\alpha}\Gamma^\alpha_{\mu\sigma} + t_{\alpha\nu}\Gamma^\alpha_{\mu\sigma} = g_{\mu\nu,\sigma} \quad 61$$

$$\Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma} + t_{\alpha\nu}g^{\alpha\lambda}\Gamma_{\lambda\mu\sigma} = g_{\mu\nu,\sigma} \quad 62$$

Solving these equations, we get the torsion applying its definition [11].

$$\Gamma_{\mu\nu\sigma} - \Gamma_{\mu\sigma\nu} = -T_{\mu\nu\sigma} \quad 63$$

Expanding the line element in powers of r^{-1} and examining the leading terms [12].

$$ds^2 = [1 - \tfrac{2M}{r} + O(r^{-2})]dt^2 + [\tfrac{4aM}{r}sin^2\theta + O(r^{-2})]dtd\phi - [1 + O(r^{-1})][dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2] \quad 64$$

Rearranging the line elements

$$ds^2 = [1 - \tfrac{2M}{r} + O(r^{-2})]dt^2 + [O(r^{-2})]dtd\phi + [\tfrac{4aM}{r}sin^2\theta] d\phi dt - [1 + O(r^{-1})][dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2] \quad 65$$

$$g_{00} = [1 - \tfrac{2M}{r} + O(r^{-2})] \quad 66$$

$$g_{03} = [O(r^{-2})] \quad 67$$

$$g_{11} = -[1 + O(r^{-1})] \quad 68$$

$$g_{22} = -[1 + O(r^{-1})]r^2 \quad 69$$

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$g_{33}=-[1+O(r^{-1})]r^2sin^2\theta$ 70

$g_{30}=[\frac{4aM}{r}sin^2\theta]$ 71

$t_{03}=[O(r^{-2})]-[\frac{4aM}{r}sin^2\theta]$ 72

$g^{00}=g_{33}(g_{00}g_{33}-g_{03}g_{30})^{-1}$ 73

$g^{03}=-g_{03}(g_{00}g_{33}-g_{03}g_{30})^{-1}$ 74

$g^{11}=g_{11}^{-1}$ 75

$g^{22}=g_{22}^{-1}$ 76

$g^{30}=-g_{30}(g_{00}g_{33}-g_{03}g_{30})^{-1}$ 77

$g^{33}=g_{00}(g_{00}g_{33}-g_{03}g_{30})^{-1}$ 78

Generalizing Clifford algebra with $\delta_{\mu\nu}^{clif}$,if $\mu\neq\nu$ and $g_{\mu\nu}$ $\neq 0$ then 1,else 0

$\{\gamma^\mu,\gamma^\nu\}=g^{\mu\nu}+g^{\nu\mu}-g^{\mu\nu}\delta_{\mu\nu}^{clif}-g^{\nu\mu}\delta_{\nu\mu}^{clif}$ 79

$\gamma^0=[1+O(r^{-1})]^{1/2}rsin\theta(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$ 80

$\gamma^1=[1+O(r^{-1})]^{-1/2}\begin{pmatrix}0 & \sigma_1\\ -\sigma_1 & 0\end{pmatrix}$ 81

$\gamma^2=[1+O(r^{-1})]^{-1/2}r^{-1}\begin{pmatrix}0 & \sigma_2\\ -\sigma_2 & 0\end{pmatrix}$ 82

$\gamma^3=[1-\frac{2M}{r}+O(r^{-2})]^{1/2}(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\begin{pmatrix}0 & \sigma_3\\ -\sigma_3 & 0\end{pmatrix}$ 83

$D^2=im\alpha E(P)-m\Sigma H(P)+im\frac{1}{2}\gamma^\mu\gamma^\nu P_\alpha T_{\mu\nu}^\alpha$, 84

from equations (16) and (17)

$F_{12}=-H_3(P)=([1+O(r^{-1})]^{-1/2})([1+O(r^{-1})]^{-1/2}r^{-1})$ $(P_{1,2}-P_{2,1})$ 85

$F_{13}=H_2(P)=([1+O(r^{-1})]^{-1/2})([1-\frac{2M}{r}+O(r^{-2})]^{1/2}$ $(g_{00}g_{33}-g_{03}g_{30})^{-1/2})(P_{1,3}-P_{3,1})$ 86

$F_{23}=-H_1(P)=([1+O(r^{-1})]^{-1/2}r^{-1})([1-\frac{2M}{r}+O(r^{-2})]$ $^{1/2}(g_{00}g_{33}-g_{03}g_{30})^{-1/2})(P_{2,3}-P_{3,2})$ 87

$F_{03}=-E_3(P)=([1+O(r^{-1})]^{1/2}rsin\theta(g_{00}g_{33}-g_{03}g_{30})^{-1/2})$ $([1-\frac{2M}{r}+O(r^{-2})]^{1/2}(g_{00}g_{33}-g_{03}g_{30})^{-1/2})(P_{0,3}-P_{3,0})$ 88

$F_{02}=-E_2(P)=([1+O(r^{-1})]^{1/2}rsin\theta(g_{00}g_{33}-g_{03}g_{30})^{-1/2})$ $([1+O(r^{-1})]^{-1/2}r^{-1})(P_{0,2}-P_{2,0})$ 89

$F_{01}=-E_1(P)=([1+O(r^{-1})]^{1/2}rsin\theta(g_{00}g_{33}-g_{03}g_{30})^{-1/2})$ $([1+O(r^{-1})]^{-1/2})(P_{0,1}-P_{1,0})$ 90

$\gamma^p=-i\gamma^{14}$ is the proyector matrix, historically γ^5, but $\gamma^5=\gamma^0\gamma^2$

$\gamma^p=-i\gamma^0\gamma^1\gamma^2\gamma^3=([1+O(r^{-1})]^{1/2}rsin\theta(g_{00}g_{33}-g_{03}g_{30})$ $^{-1/2})([1+O(r^{-1})]^{-1/2})([1+O(r^{-1})]^{-1/2}r^{-1})([1-\frac{2M}{r}+O$ $(r^{-2})]^{1/2}(g_{00}g_{33}-g_{03}g_{30})^{-1/2})\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix}$ 91

Geodesic Equation and Torsion Tensor

A geodesic that is not a null geodesic has the property that $\int ds$, taken along a section of the track with the end points P and Q, is stationary if one makes a small variation of the track keeping the end points fixed. If dx^μ denotes an element along the track [13].

$ds^2=g_{\mu\nu}dx^\mu dx^\nu$ 92

$2ds\delta(ds)=dx^\mu dx^\nu\delta(g_{\mu\nu})+g_{\mu\nu}\delta(dx^\mu)dx^\nu+g_{\mu\nu}dx^\mu\delta(dx^\nu)$ 93

$2ds\delta(ds)=dx^\mu dx^\nu g_{\mu\nu},_\lambda\delta(x^\lambda)+2g_{\mu\lambda}dx^\mu\delta(dx^\lambda)$ 94

$\delta(dx^\lambda)=d\delta(x^\lambda)$ and $dx^\mu=v^\mu ds$ 95

$\int\delta(ds)=\int[\frac{1}{2}g_{\mu\nu},_\lambda v^\mu v^\nu\delta x^\lambda+g_{\mu\lambda}v^\mu\frac{d\delta x^\lambda}{ds}]ds$ 96

By partial integration with $\delta x^\lambda=0$ at end points P and Q, we get

$\delta\int ds=\int[\frac{1}{2}g_{\mu\nu},_\lambda v^\mu v^\nu-\frac{d}{ds}(g_{\mu\lambda}v^\mu)]\delta x^\lambda ds$ 97

The condition for this to vanish with arbitrary δx^λ is

$\frac{d}{ds}(g_{\mu\lambda}v^\mu)-\frac{1}{2}g_{\mu\nu},_\lambda v^\mu v^\nu=0$ 98

$$\frac{d}{ds}(g_{\mu\lambda}v^{\mu}) = g_{\mu\lambda}\frac{dv^{\mu}}{ds} + g_{\mu\lambda,\nu}\, v^{\mu}v^{\nu} \quad 99$$

$\frac{d}{ds}(g_{\mu\lambda}v^{\mu}) = g_{\lambda\mu}\frac{dv^{\mu}}{ds} - t_{\lambda\mu}\frac{dv^{\mu}}{ds} + \frac{1}{2}(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu})v^{\mu}v^{\nu}$, and

with $t_{\mu\lambda} = g_{\mu\lambda} - g_{\lambda\mu}$ 100

$$\frac{d}{ds}(g_{\mu\lambda}v^{\mu}) = g_{\lambda\mu}\frac{dv^{\mu}}{ds} - t_{\lambda\mu}\frac{dv^{\mu}}{ds} + \frac{1}{2}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu})v^{\mu}v^{\nu} + \frac{1}{2}$$

$$(t_{\mu\lambda,\nu} + t_{\nu\lambda,\mu})v^{\mu}v^{\nu} \quad 101$$

From equation (98) with $t_{\mu\nu,\lambda} = g_{\mu\nu,\lambda} - g_{\nu\mu,\lambda}$

$$\frac{d}{ds}(g_{\mu\lambda}v^{\mu}) - \frac{1}{2}g_{\nu\mu,\lambda}\, v^{\mu}v^{\nu} - \frac{1}{2}t_{\mu\nu,\lambda}\, v^{\mu}v^{\nu} = 0 \quad 102$$

Thus the condition (102) becomes

$$g_{\lambda\mu}\frac{dv^{\mu}}{ds} - t_{\lambda\mu}\frac{dv^{\mu}}{ds} + \bar{\Gamma}_{\lambda\mu\nu}v^{\mu}v^{\nu} - \dot{\Gamma}_{\lambda\mu\nu}v^{\mu}v^{\nu} = 0 \quad 103$$

$$\bar{\Gamma}_{\lambda\mu\nu} = \frac{1}{2}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}) \quad 104$$

$$\dot{\Gamma}_{\lambda\mu\nu} = \frac{1}{2}(t_{\lambda\mu,\nu} + t_{\lambda\nu,\mu} - t_{\nu\mu,\lambda}) \quad 105$$

Multiplying equation (103) by $g^{\sigma\lambda}$, we obtain the geodesic equation

$$\frac{dv^{\sigma}}{ds} - g^{\sigma\lambda}t_{\lambda\mu}\frac{dv^{\mu}}{ds} + \bar{\Gamma}^{\sigma}_{\mu\nu}v^{\mu}v^{\nu} - \dot{\Gamma}^{\sigma}_{\mu\nu}v^{\mu}v^{\nu} = 0 \quad 106$$

$$\frac{dv^{\sigma}}{ds} - g^{\sigma\lambda}t_{\lambda\mu}\frac{dv^{\mu}}{ds} + \Gamma^{\sigma}_{\mu\nu}v^{\mu}v^{\nu} = 0 \quad 107$$

$$\Gamma^{\sigma}_{\mu\nu} = \bar{\Gamma}^{\sigma}_{\mu\nu} - \dot{\Gamma}^{\sigma}_{\mu\nu} \quad 108$$

$\bar{\Gamma}^{\sigma}_{\mu\nu}$ are the Christoffel symbols of the symmetric part, so

$$-\bar{T}^{\sigma}_{\mu\nu} = \bar{\Gamma}^{\sigma}_{\mu\nu} - \bar{\Gamma}^{\sigma}_{\nu\mu} = \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu} + \dot{\Gamma}^{\sigma}_{\mu\nu} - \dot{\Gamma}^{\sigma}_{\nu\mu} \quad 109$$

$$0 = \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu} + \dot{\Gamma}^{\sigma}_{\mu\nu} - \dot{\Gamma}^{\sigma}_{\nu\mu} \quad 110$$

We directly obtain the torsion tensor without solving equations (62) and (63)

$$T^{\sigma}_{\mu\nu} = \dot{\Gamma}^{\sigma}_{\mu\nu} - \dot{\Gamma}^{\sigma}_{\nu\mu} \quad 111$$

Einstein Field Equation and Conservation Laws

From equation (108) where $\bar{\Gamma}^{\sigma}_{\mu\nu}$ are the symbols of the symmetric part

$$\Gamma^{\mu}_{\nu\mu} = \bar{\Gamma}^{\mu}_{\nu\mu} - \dot{\Gamma}^{\mu}_{\nu\mu} \quad 112$$

$$\bar{\Gamma}^{\mu}_{\nu\mu} = \bar{\Gamma}^{\mu}_{\mu\nu} = g^{\mu\lambda}\bar{\Gamma}_{\lambda\mu\nu} = g^{\mu\lambda}\frac{1}{2}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}) \quad 113$$

$$\bar{\Gamma}^{\mu}_{\nu\mu} = (\sqrt{-g})^{-1}\sqrt{-g}_{,\nu} + g^{\mu\lambda}\frac{1}{2}(g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}) \quad 114$$

Equation (112) becomes

$$\Gamma^{\mu}_{\nu\mu} = (\sqrt{-g})^{-1}\sqrt{-g}_{,\nu} + g^{\mu\lambda}\frac{1}{2}(g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}) - \dot{\Gamma}^{\mu}_{\nu\mu} \quad 115$$

$$\Gamma^{\mu}_{\nu\mu} = (\sqrt{-g})^{-1}\sqrt{-g}_{,\nu} - t_{\nu} \quad 116$$

$$t_{\nu} = -g^{\mu\lambda}\frac{1}{2}(g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}) + \dot{\Gamma}^{\mu}_{\nu\mu} \quad 117$$

The vector A^{μ} has the covariant divergence

$$A^{\mu}{}_{;\mu} = A^{\mu}{}_{,\mu} + \Gamma^{\mu}_{\nu\mu}A^{\nu} \quad 118$$

$$(A^{\mu}{}_{;\mu} + t_{\nu}A^{\nu})\sqrt{-g} = (A^{\mu}\sqrt{-g})_{,\mu} \quad 119$$

If the left-hand side of equation (119) equals zero then the right-hand side gives us the first conservation law.

For the antisymmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$

$$F^{\mu\nu}{}_{;\nu} = F^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\rho\nu}F^{\rho\nu} + \Gamma^{\nu}_{\rho\nu}F^{\mu\rho} \quad 120$$

$$F^{\mu\nu}{}_{;\nu} = F^{\mu\nu}{}_{,\nu} - T^{\mu}_{\rho\nu}F^{\rho\nu} + ((\sqrt{-g})^{-1}\sqrt{-g}_{,\rho} - t_{\rho})F^{\mu\rho} \quad 121$$

$$(F^{\mu\nu}{}_{;\nu} + T^{\mu}_{\rho\nu}F^{\rho\nu} + t_{\rho}F^{\mu\rho})\sqrt{-g} = (F^{\mu\nu}\sqrt{-g})_{,\nu} \quad 122$$

If the left-hand side of equation (122) equals zero then the right-hand side gives us the second conservation law.

For the antisymmetric tensor $F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu}$

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} + T^{\alpha}_{\mu\nu}A_{\alpha} \quad 123$$

$$F_{\mu\nu;\sigma} = F_{\mu\nu,\sigma} - \Gamma^{\alpha}_{\mu\sigma}F_{\alpha\nu} - \Gamma^{\alpha}_{\nu\sigma}F_{\mu\alpha} \quad 124$$

$$F_{\nu\sigma;\mu} = F_{\nu\sigma,\mu} - \Gamma^{\alpha}_{\nu\mu}F_{\alpha\sigma} - \Gamma^{\alpha}_{\sigma\mu}F_{\nu\alpha} \quad 125$$

$$F_{\sigma\mu;\nu} = F_{\sigma\mu,\nu} - \Gamma^{\alpha}_{\sigma\nu}F_{\alpha\mu} - \Gamma^{\alpha}_{\mu\nu}F_{\sigma\alpha} \quad 126$$

Adding equations (124), (125) and (126)

$$F_{\mu\nu;\sigma} + F_{\nu\sigma;\mu} + F_{\sigma\mu;\nu} = T^{\alpha}_{\mu\nu}F_{\sigma\alpha} + (T^{\alpha}_{\mu\nu}A_{\alpha})_{,\sigma} + T^{\alpha}_{\nu\sigma}F_{\mu\alpha}$$

$$+ (T^{\alpha}_{\nu\sigma}A_{\alpha})_{,\mu} + T^{\alpha}_{\sigma\mu}F_{\nu\alpha} + (T^{\alpha}_{\sigma\mu}A_{\alpha})_{,\nu} \quad 127$$

From the definition of the curvature tensor $R^{\beta}_{\nu\rho\sigma}$

$$R^{\beta}_{\nu\rho\sigma} = \Gamma^{\beta}_{\nu\sigma,\rho} - \Gamma^{\beta}_{\nu\rho,\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\beta}_{\alpha\sigma} \quad 128$$

$R^{\mu}_{\nu\mu\rho}$ is called the Ricci tensor

$$R_{\mu\nu} = -\Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} \qquad 129$$

Now $R_{\mu\nu}$ is not symmetric, $2A_{\mu\nu} = R_{\mu\nu} - R_{\nu\mu}$ is the antisymmetric part and $R_{\mu\nu} = A_{\mu\nu} + S_{\mu\nu}$ where $S_{\mu\nu}$ is the symmetric part in the Einstein's equation [14].

$$S_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}S = \kappa T_{\mu\nu} \qquad 130$$

$$2A_{\mu\nu} = -((\sqrt{-g})^{-1}\sqrt{-g}_{,\mu} - t_{\mu})_{,\nu} + ((\sqrt{-g})^{-1}\sqrt{-g}_{,\nu} - t_{\nu}),$$

$$_{\mu} - T^{\alpha}_{\mu\nu,\alpha} - T^{\alpha}_{\mu\nu}((\sqrt{-g})^{-1}\sqrt{-g}_{,\alpha} - t_{\alpha}) \qquad 131$$

$$2A_{\mu\nu} = t_{\mu,\nu} - t_{\nu,\mu} - T^{\alpha}_{\mu\nu,\alpha} - T^{\alpha}_{\mu\nu}((\sqrt{-g})^{-1}\sqrt{-g}_{,\alpha} - t_{\alpha}) \qquad 132$$

$$R_{\mu\nu} - A_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}(R - A) = \kappa T_{\mu\nu} \qquad 133$$

$$R_{\mu\nu} - A_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \qquad 134$$

Conclusions

Multilevel operator D^{mul} $(g_{\mu\nu}, P_{\mu}, q)$ has be generalized for a curved space with a general four-potential P. For gravity $G_{\mu\nu}$ (P) is the new gravitomagnetic tensor and torsion tensor $T^{\propto}_{\mu\nu}$ appears in its definition.

In a at space $G_{\mu\nu}$ $(A) = F_{\mu\nu}$ (A), D3 and D4 operators vanish. In a curved space the curvature tensor $R^{\propto}_{\mu\nu\delta}$ appears in levels 3 and 4.

The appearance of torsion tensor $T^{\propto}_{\mu\nu}$ and curvature tensor $R^{\propto}_{\mu\nu\delta}$ in multilevel operator $D^{mul}$$(g_{\mu\nu}, P_{\mu}, q)$ means that this operator is a fundamental operator in Quantum Field Theory.

$\gamma^{0}, \gamma^{1}, \gamma^{2}, \gamma^{3}$, have been calculated for Schwarzschild's metric, then $G_{\mu\nu}$ (P), the gravitomagnetic tensor has been obtained.

Each D^{n}, where n is the number of γ matrices in the product of the algebra members, is related to an n-form.

The invariance of the length of vectors under parallel transport requires the vanishing of the metric tensor covariant derivative, a new term appears in equation (59) with $t_{\mu\nu}=g_{\mu\nu}-g_{\nu\mu}$ measuring the non-symmetric part of the metric tensor, solving these equations we get the torsion tensor.

Rearranging Kerr's metric we obtained $t03 = g03 - g30$, the non-symmetric part of the metric tensor, gravitomagnetic tensor has also been calculated generalizing the Clifford algebra.

Taking into account the $t_{\mu\nu}=g_{\mu\nu}-g_{\nu\mu}$ in the geodesic equation we have obtained the torsion tensor, conservation laws and Einstein field equation in a non-symmetric geometry.

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