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Short Communication

On Twisted Space

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Abstract

In a manifold with a linear connection if the torsion tensor vanishes then the Christoffel symbols are symmetric in the lower indices, this is the motivation to call it the symmetric connection.

The requirement of being metric, the length is preserved by parallel transport and symmetric leads to the unique result for the Christoffel symbols of the connection.

This distinguished linear connection on a Riemannian manifold is usually called the Riemann connection or the Levi-Civita connection.

We consider a connection which is metric but not symmetric, the torsion tensor does not vanish and we study the conservation laws depending on it, the non-symmetric Ricci tensor and the action.

Keywords

Twisted Space; Torsion Tensor; Conservation Law; Curved Space; General Relativity; Riemann Curvature Tensor; Ricci Tensor

Introduction

Galaxies in our universe are rotating with such speed that the gravity generated by their observable matter could not possibly hold them together; an extra force is needed to explain it.

In a galaxy there are no dark matter particles generating the dark matter force to explain this extra force, new terms in the action depending on the torsion explain this extra force.

This new force also defines the distribution of galaxies in the universe.

A metric connection not symmetric

The Christoffel symbols of the first kind of the connection with a given torsion T satisfy [1].

$$g_{ij,k} = \Gamma_{ijk} + \Gamma_{jik} \tag{1}$$

$$-T_{ijk} = \Gamma_{ijk} - \Gamma_{ikj} \tag{2}$$

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l}) - \tfrac{1}{2}(T_{jk}{}^{i} + T_{kj}{}^{i} + T^{i}{}_{jk}) \tag{3}$$

Differentiating the determinant g [2] and defining

$$T_j = \tfrac{1}{2}(T_{ji}{}^{i} + T_{ij}{}^{i} + T^{i}{}_{ji})$$

$$\Gamma^i_{ji} = (\sqrt{-g})^{-1}\sqrt{-g}_{,j} - T_j \tag{4}$$

The vector A^μ has the covariant divergence [3].

$$A^\mu{}_{;\mu} = A^\mu{}_{,\mu} + \Gamma^\mu_{\nu\mu} A^\nu \tag{5}$$

$$(A^\mu{}_{;\mu} + T_\nu A^\nu)\sqrt{-g} = (A^\mu \sqrt{-g})_{,\mu} \tag{6}$$

If the left-hand side of equation (6) equals zero then the right-hand side gives us the first conservation law.

For the antisymmetric tensor $F^{\mu\nu}$=-$F^{\nu\mu}$.

$$F^{\mu\nu}{}_{;\nu} = F^{\mu\nu}{}_{,\nu} + \Gamma^\mu_{\rho\nu} F^{\rho\nu} + \Gamma^\nu_{\rho\nu} F^{\mu\rho} \tag{7}$$

$$F^{\mu\nu}{}_{;\nu} = F^{\mu\nu}{}_{,\nu} - T^\mu_{\rho\nu} F^{\rho\nu} + ((\sqrt{-g})^{-1}\sqrt{-g}_{,\rho} - T_\rho) F^{\mu\rho} \tag{8}$$

$$(F^{\mu\nu}{}_{;\nu} + T^\mu_{\rho\nu} F^{\rho\nu} + T_\rho F^{\mu\rho})\sqrt{-g} = (F^{\mu\nu}\sqrt{-g})_{,\nu} \tag{9}$$

If the left-hand side of equation (9) equals zero then the right-hand side gives us the second conservation law [4].

In the symmetric case $Y^{\mu\nu}$=$Y^{\nu\mu}$ we can get a corresponding equation, provided we put one of the suffixes downstairs and deal with $Y^{\mu\nu}{}_{;\nu}$

$$Y^{\nu}_{\mu;\nu} = Y^{\nu}_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu} Y^{\nu}_{\alpha} + \Gamma^{\nu}_{\alpha\nu} Y^{\alpha}_{\mu} \tag{10}$$

$$Y^{\nu}_{\mu;\nu} = Y^{\nu}_{\mu,\nu} - \Gamma_{\alpha\mu\nu} Y^{\alpha\nu} + ((\sqrt{-g})^{-1}\sqrt{-g}_{,\alpha} - T_{\alpha}) Y^{\alpha}_{\mu} \tag{11}$$

Since $Y^{\mu\nu}=Y^{\nu\mu}$ is symmetric, we can replace the $\Gamma_{\alpha\mu\nu}$ from (1) by

$$\tfrac{1}{2}(\Gamma_{\alpha\nu\mu} + \Gamma_{\nu\alpha\mu}) = \tfrac{1}{2} g_{\alpha\nu,\mu} \tag{12}$$

$$(Y^{\nu}_{\mu;\nu} + \tfrac{1}{2} g_{\alpha\beta,\mu} Y^{\alpha\beta} + T_{\alpha} Y^{\alpha}_{\mu})\sqrt{-g} = (Y^{\nu}_{\mu}\sqrt{-g})_{,\nu} \tag{13}$$

For the antisymmetric tensor $F_{\mu\nu}=A_{\mu;\nu} - A_{\nu;\mu}$

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} + T^{\alpha}_{\mu\nu} A_{\alpha} \tag{14}$$

$$F_{\mu\nu\,;\sigma} = F_{\mu\nu\,,\sigma} - \Gamma^{\alpha}_{\mu\sigma} F_{\alpha\nu} - \Gamma^{\alpha}_{\nu\sigma} F_{\mu\alpha} \tag{15}$$

$$F_{\nu\sigma\,;\mu} = F_{\nu\sigma\,,\mu} - \Gamma^{\alpha}_{\nu\mu} F_{\alpha\sigma} - \Gamma^{\alpha}_{\sigma\mu} F_{\nu\alpha} \tag{16}$$

$$F_{\sigma\mu\,;\nu} = F_{\sigma\mu\,,\nu} - \Gamma^{\alpha}_{\sigma\nu} F_{\alpha\mu} - \Gamma^{\alpha}_{\mu\nu} F_{\sigma\alpha} \tag{17}$$

Adding equations (15), (16) and (17)

$$F_{\mu\nu\,;\sigma} + F_{\nu\sigma\,;\mu} + F_{\sigma\mu\,;\nu} = T^{\alpha}_{\mu\nu} F_{\sigma\alpha} + (T^{\alpha}_{\mu\nu} A_{\alpha})_{,\sigma} + T^{\alpha}_{\nu\sigma} F_{\mu\alpha} + (T^{\alpha}_{\nu\sigma} A_{\alpha})_{,\mu} + T^{\alpha}_{\sigma\mu} F_{\nu\alpha} + (T^{\alpha}_{\sigma\mu} A_{\alpha})_{,\nu} \tag{18}$$

From the definition of the curvature tensor $R^{\beta}{}_{\nu\rho\sigma}$ [5].

$$R^{\beta}_{\nu\rho\sigma} = \Gamma^{\beta}_{\nu\sigma,\rho} - \Gamma^{\beta}_{\nu\rho,\sigma} + \Gamma^{\alpha}_{\nu\sigma} \Gamma^{\beta}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho} \Gamma^{\beta}_{\alpha\sigma} \tag{19}$$

$R^{\mu}{}_{\nu\rho\mu}$ is called the Ricci tensor [6].

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu} \Gamma^{\beta}_{\alpha\beta} \tag{20}$$

Now $R_{\mu\nu}$ is not symmetric, $A_{\mu\nu}=R_{\mu\nu}-R_{\nu\mu}$ is the antisymmetric part and $2R_{\mu\nu}=A_{\mu\nu}+S_{\mu\nu}$ where $S_{\mu\nu}$ is the symmetric part in the Einstein's equation.

$$A_{\mu\nu} = ((\sqrt{-g})^{-1}\sqrt{-g}_{,\mu} - T_{\mu})_{,\nu} - ((\sqrt{-g})^{-1}\sqrt{-g}_{,\nu} - T_{\nu})_{,\mu} + T^{\alpha}_{\mu\nu,\alpha} + T^{\alpha}_{\mu\nu}((\sqrt{-g})^{-1}\sqrt{-g}_{,\alpha} - T_{\alpha}) \tag{21}$$

The δI for the antisymmetric tensor $F_{\mu\nu}=A_{\mu;\nu}-A_{\nu;\mu}$ [7].

$$\delta I = \int (-\tfrac{1}{2} E^{\mu\nu} \delta g_{\mu\nu} + (4\pi)^{-1} (F^{\mu\nu}{}_{;\nu} + T^{\mu}_{\rho\nu} F^{\rho\nu} + T_{\rho} F^{\mu\rho}) \delta A_{\mu}) \sqrt{-g} \tag{22}$$

where $E^{\rho\sigma}$ is the stress-energy tensor of the field [7].

$$E^{\rho\sigma} = (4\pi)^{-1} (-F^{\rho}_{\nu} F^{\sigma\nu} + \tfrac{1}{4} g^{\rho\sigma} F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} \delta(T^{\gamma}_{\alpha\beta} A_{\gamma}) g^{\mu\alpha} g^{\nu\beta}) \tag{23}$$

The term, $F_{\mu\nu}\ \delta\ (T^{\gamma}_{\alpha\beta}A_{\gamma})\ g^{\mu\alpha}\ g^{\nu\beta}$, is derived from the new dependence of $F_{\mu\nu}$ on torsion tensor and the dependence of $T_{\mu\nu}$ on $g_{\mu\nu}$.

The other new terms depending on $T_{\mu\nu}$ are derived from equation (9).

Conclusions

Tensor densities have been defined leading to conservation laws depending on the torsion, if torsion is zero these equations return to the known conservation laws in General Relativity.

The antisymmetric part of the Ricci tensor has been defined depending on the torsion tensor; if torsion tensor vanishes the Ricci tensor is symmetric and leads to Einstein's equation. Action has been defined and there are new terms depending on the torsion, if the torsion tensor vanishes this equation return to the known action.

Space must also be twisted, for a complete image of the universe we need the antisymmetric part of the connection.

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