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On Planck's Gravitational Constant and Kepler's Third Law Derived from Schrödinger's Equation

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Abstract

The solution of Schrödinger's equation for a gravitational system leads us to Planck's gravitational constant and Kepler's third law.The solution of Schrödinger's equation for an electron orbiting a nucleus leads us to the emission of gravitomagnetic photons when the gravitational potential is also taken into account. In four dimensions, the Minkowski metric $\eta_{\mu\nu}$= diag (+1,–1,–1,–1) leads to the 16-dimensional Clifford algebra C(1,3), Dirac's equation [1] is using four of these 16 matrices that form a basis of this algebra, a new operator is defined using all of these matrices and also generalized for a curved space. This new multilevel operator generalizes the Dirac's equation, the value of the generalized Dirac's operator is calculated in the Schwarzschild's metric. The torsion tensor is calculated taking into account the non-symmetric part of the metric tensor in the vanishing of its covariant derivative and applied to Kerr's metric generalizing the Clifford algebra. Geodesic equation, conservation laws, torsion tensor and Einstein field equation are obtained in a non-symmetric geometry.

Keywords: *Planck's Gravitational Constant, Planck's Constant, Kepler's Third Law, Schrödinger Equation; Gravitomagnetic Photon Emission; Gravitomagnetic Tensor; Gravitational Magnetic Field; Energy-Momentum 1-Form; Clifford Algebra; Dirac Equation; Dirac Operator; Gravity and Quantum Mechanics Unification; Multilevel Operator; Schwarzschild's Metric; Torsion Tensor; Rearranged Kerr's Metric; Generalized Dirac Equation; Generalized Clifford Algebra; Generalized Einstein Field Equation; Generalized Geodesic Equation; Conservation Laws; Non-Symmetric Geometry*

Introduction

Dirac's equation is the relativistic wave equation derived by physicist Paul Dirac in 1928. The wave functions in the Dirac theory are vectors of four complex components (known as bispinors), two of which resemble the Pauli wavefunction in the non-relativistic limit, in contrast to the Schrödinger equation which described wave functions of only one complex component.

Dirac's operator is just the tip of the iceberg, the tip of a generalized operator that is obtained by operating on all members of the Clifford algebra basis and not just on four of them.

The Schwarzschild's metric is named in honour of Karl Schwarzschild, who found the exact solution in 1915 and published it in January 1916, a little more than a month after the publication of Einstein's theory of general relativity. It was the first exact solution of the Einstein field equations other than the trivial flat space solution. Schwarzschild died shortly after his paper was published, as a result of a disease he developed while serving in the German army during World War I. Johannes Droste in 1916 independently produced the same solution as Schwarzschild.

Schwarzschild's metric is an exact solution to the Einstein's field equations that describes the gravitational field outside a spherical mass, on the assumption that the electric charge of the mass, angular momentum of the mass, and universal cosmological constant are all zero.

The new generalized Dirac's operator, the multilevel operator, is calculated in the Schwarzschild's metric, torsion tensor and new gravitomagnetic tensor appear in level 2, curvature tensor appears in levels 3 and 4.

The Kerr's metric is a generalization to a rotating body of the Schwarzschild's metric. The Einstein field equation relates the geometry of spacetime to the distribution of matter within it. The equations were published by Einstein in 1915 in the form of a tensor equation which related the local spacetime curvature with the local energy, momentum and stress within that spacetime expressed by the stress-energy tensor.

Planck's gravitational constant

In the hydrogen atom an electron is orbiting a nucleus with 1 proton, we know the energy levels from the solution of the Schrödinger's equation[2], where m_p is the proton mass, m_e is the electron mass, μ is the 2-body reduced mass, e is the electron charge, r is the position of the electron relative to the nucleus, the potential term is due to the Coulomb interaction wherein ϵ_0 is the permittivity of free space.

$$\mu = \frac{m_p m_e}{m_p + m_e} \tag{1}$$

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} = -\frac{k_e}{r} \tag{2}$$

$$E_n = -\frac{\mu e^4}{(4\pi\epsilon_0)^2 2\hbar^2 n^2} \tag{3}$$

$$E_n = -\frac{\mu k^2}{2\hbar^2 n^2} \tag{4}$$

with $k = k_e$, now for a gravitational potential

$$V(r) = -\frac{k_g}{r} \tag{5}$$

And in equation (4) we replace $\hbar$ by $\hbar_g$

$$E_n = -\frac{\mu k_g^2}{2\hbar_g^2 n^2} \tag{6}$$

Planck's gravitational constant values

We apply equation (6) to the Sun-Earth system, equating equation (6) to the total energy of the gravitational system we get the value of Planck's gravitational constant in this system, $M_s = 1.9885 \cdot 10^{30}$ kg is the Sun mass, $M_e = 5.97237 \cdot 10^{24}$ kg is the Earth mass, $a = 149598023000$ m is the semi-major axis, eccentricity $e = 0.0167086$, $k_g = GM_s\mu$, $n = l + 1$ and L the angular momentum. The total energy of the gravitational system is defined by

$$E = -\frac{GM_s\mu}{2a} = -\frac{k_g}{2a} \tag{7}$$

$$\mu = \frac{M_s M_e}{M_s + M_e} \tag{8}$$

$$L^2 = l(l+1)\hbar_g^2 = GM_s a\mu^2(1 - e^2) \tag{9}$$

$$n^2\hbar^2 = (l+1)^2\hbar_g^2 = GM_s a\mu^2 \tag{10}$$

$$\frac{l}{l+1} = \frac{n-1}{n} = 1 - e^2, n = e^{-2} \text{ from } (9)/(10) \tag{11}$$

$$n = 3581.9529381362201856 \tag{12}$$

$$n = 3582, e = 0.0167084902372362 \tag{13}$$

$$E = -\frac{\mu k_g^2}{2\hbar_g^2 n^2} = -\frac{k_g}{2a} \tag{14}$$

$$\hbar_g^2 = \frac{\mu k_g a}{n^2} \tag{15}$$

$$\hbar_g = 7.4290571574528236410479940684343716337342817210645879404396660 45 \cdot 10^{36} \tag{16}$$

$$h_g = 4.66781427779049252488785364223568563336168774518864147659109445351615 \cdot 10^{37} \tag{17}$$

We apply equation (6) to the Sun-Jupiter system, $M_s = 1.9885 \cdot 10^{30}$ kg is the Sun mass, $M_j = 1.8982 \cdot 10^{27}$ kg is the Jupiter mass, $a = 778547261754.2769$ m is the semi-major axis, eccentricity $e = 0.04839266$, $k_g = GM_s\mu$

$$n = 427.0129152699940647 \tag{18}$$

$$n = 427, e = 0.0483933918495827 \quad (19)$$

$$\mu = \frac{M_s M_j}{M_s + M_j} \quad (20)$$

$$E = -\frac{\mu k_g^2}{2\hbar_g^2 n^2} = -\frac{k_g}{2a} \quad (21)$$

$$\hbar_g^2 = \frac{\mu k_g a}{n^2} \quad (22)$$

$$\hbar_g = 4.5143285454267936940204253931854795356384121809605079010026548501179 4 \cdot 10^{40} \quad (23)$$

$$h_g = 2.8364362788407024541545842846823831136665704506872014999605746678517 25 \cdot 10^{41} \quad (24)$$

Now we consider the hydrogen atom with $n = 1$, from equations (3) and (4)

$$E = -\frac{\mu k_e^2}{2\hbar^2} = -\frac{k_e}{2a_0} \quad (25)$$

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2} = 0.52946540982610384737791092392484241870860135147716193393537974 88 \cdot 10^{-10} \quad (26)$$

Comparing equations (25) and (21) we see the role played by the semi-major axis a now is played by a_0 and from equation (22)

$$\hbar_g^2 = \mu k_g a_0 \quad (27)$$

$$\hbar_g = 2.2133922090705168884053659412952213843762826013967180401998187912 5 \cdot 10^{-54} \quad (28)$$

$$h_g = 1.3907153407057639149004805826049317930723381117551132578366994666 38 \cdot 10^{-53} \quad (29)$$

Now if we adapt equation (27) to k_e we should obtain the Planck's constant value

$$\hbar^2 = \mu k_e a_0 \quad (30)$$

$$\hbar = 1.0545718176461564 \cdot 10^{-34} \quad (31)$$

$$h = 6.62607015000000000549 \cdot 10^{-34} \quad (32)$$

Kepler's third law

If the Hamiltonian is not an explicit function of time, the wave function is separable into a product of spatial and temporal parts [3].

$$\psi(r,t) = \psi(r) e^{\frac{-iEt}{\hbar}} \quad (33)$$

T is the period and Kepler's third law is defined by

$$\frac{GM_s}{a^3} = \frac{4\pi^2}{T^2} \quad (34)$$

E is the total Energy in a gravitational system defined by $E = \frac{GM_s\mu}{2a}$ (35)

From equation (10)

$$\hbar = \frac{(GM_s a)^{\frac{1}{2}}\mu}{n} \quad (36)$$

and

$$\frac{E}{\hbar} = \frac{GM_s\mu}{2a}\frac{n}{(GM_s a)^{\frac{1}{2}}\mu} = \frac{n}{2}\frac{(GM_s)^{\frac{1}{2}}}{a^{\frac{3}{2}}} = \frac{n}{2}\omega_m \quad (37)$$

ω_m is the mean motion angular speed defined by

$$\omega_m = \frac{2\pi}{T} \tag{38}$$

$$\omega_m = \frac{(GM_s)^{\frac{1}{2}}}{a^{\frac{3}{2}}} \tag{39}$$

equations (38) and (39) define Kepler's third law and

$$\psi(r,t) = \psi(r)e^{\frac{-in\omega_m t}{2}} \tag{40}$$

Five new planets in Proxima Centauri

Mean motion angular speed ω_m for our planets:

ω_{m1} = 0:0000008266683161721671725893680342060 – *Merucry*

ω_{m2} = 0:0000003236397806290027502923891805337 – *Venus*

ω_{m3} = 0:0000001990958336720942466833404885350 – *Earth*

ω_{m4} = 0:0000001058577386399185014918267545470 – *Mars*

ω_{m5} = 0:00000004324349662 – *Ceres*

ω_{m6} = 0:000000017320508 –*Jupiter*

ω_{m7} = 0:0000000067118273148381163645269302 – *Saturn*

ω_{m8} = 0:0000000023610970045003705167333373453 – *Uranus*

ω_{m9} = 0:0000000012054073971413942728010767548 – *Netpune*

ω_{m10} = 0:0000000008092269920779060844908523775485 – *Pluto* (41)

ω_{m1}/ω_{mn} ratios:

$\omega_{m1}/\omega_{m2} = 2.5542852444329146707793372814101859983\,42$

$\omega_{m1}/\omega_{m3} = 4.152112582796025641132599537390429110737$

$\omega_{m1}/\omega_{m4} = 7.809238387229576415481006972503699305849$

$\omega_{m1}/\omega_{m5} = 19.116592800912295250683362390087871668505$

$\omega_{m1}/\omega_{m6} = 47.727717695818573715584325483178668893545$

$\omega_{m1}/\omega_{m7} = 123.16590957944002434443145227616616508490\,29$

$\omega_{m1}/\omega_{m8} = 350.12043749007155504550324403124830431920\,0979$

$\omega_{m1}/\omega_{m9} = 685.79993629755284246820561633365273445660675$

$\omega_{m1}/\omega_{m10} = 1021.5530676373953341760499255797110784478856049$ (42)

Mean motion angular speed ω_m for planets in Proxima Centauri:

$\omega_{m1} = 0.00001408714662378\,4$ –Proxima–d
$\omega_{m2} = 0.000006513393892588$ –Proxima –b
$\omega_{m3} = 0.0000000398204257868$ –Proxima – c (43)

ω_{m1}/ω_{mn} ratios:

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$$\omega_{m1}/\omega_{m2} = 2.162796670383261930601$$

$$\omega_{m1}/\omega_{m3} = 353.76685069132842946962 \quad (44)$$

Comparing equations (42) and (44) we see a gap for 5 planets from ω_{m1}/ω_{m3} to ω_{m1}/ω_{m7}

Gravitomagnetic photon emission

An electron is orbiting a nucleus with Z protons, we know the energy levels from the solution of the Schrödinger equation[2], where m_p is the proton mass, m_e is the electron mass, μ is the 2-body reduced mass, e is the electron charge, r is the position of the electron relative to the nucleus, the potential term is due to the Coulomb interaction wherein ϵ_0 is the permittivity of free space and m_N is the mass of the nucleus.

$$\mu = \frac{m_N m_e}{m_N + m_e} \quad (45)$$

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{k_e}{r} \quad (46)$$

$$E_n = -\frac{\mu Z^2 e^4}{(4\pi\epsilon_0)^2 2\hbar^2 n^2} \quad (47)$$

$$E_n = -\frac{\mu k^2}{2\hbar^2 n^2} \quad (48)$$

$$C_n = \frac{\mu}{2\hbar^2 n^2} \quad (49)$$

$$E_n = -C_n k^2 \quad (50)$$

with $k = k_e$, now adding the gravitational potential

$$V(r) = -\frac{k_e}{r} - \frac{G m_N \mu}{r} = -\frac{k_e}{r} - \frac{k_g}{r} = -\frac{(k_e + k_g)}{r} = -\frac{k}{r} \quad (51)$$

And from equation (50)

$$E_n = -C_n\left(k_e^2 + 2k_e k_g + k_g^2\right) \quad (52)$$

$$E_n = -C_n\left(k_e^2 + k_e k_g + k_g^2 + k_g k_e\right) \quad (53)$$

$$E_n^e = -C_n\left(k_e^2 + k_e k_g\right) \quad (54)$$

$$E_n^g = -C_n\left(k_g^2 + k_g k_e\right) \quad (55)$$

$$E_n = E_n^e + E_n^g \quad (56)$$

$$h\nu_n^e = \left(C_n^f - C_n^i\right)\left(k_e^2 + k_e k_g\right) \quad (57)$$

$$h_g \nu_n^g = \left(C_n^f - C_n^i\right)\left(k_g^2 + k_g k_e\right) \quad (58)$$

From equation (58) ν_n^g is the frequency of the gravitomagnetic photon emitted from the initial energy level to the final energy level. This emission leads us to the gravitomagnetic tensor. Gravitational magnetic field generates the extra force needed to explain the anomalous behavior of pendulums observed during a solar eclipse, the Allais effect [5] and also explains the dark matter effect without exotic particles never detected. Gravitational magnetic field is also derived from Special Relativity force transformations [6], when velocities point to the same direction a repulsive gravitational magnetic force is induced. Gravitomagnetic tensor will appear below in equation (88) at level two of the generalized Dirac equation D^2

From equation (57) ν_n^e is the frequency of the electromagnetic photon emitted from the initial energy level to the final energy level. The correction of the second term is an indirect detection of the gravitomagnetic photon emission

$$\alpha = \frac{e^2}{(4\pi\epsilon_0)\hbar c} \quad (59)$$

$$E = -\frac{\mu Z^2 e^4}{(4\pi\epsilon_0)^2 2\hbar^2 n^2}\left[1 + \frac{Z^2\alpha^2}{n}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right)\right] \quad (60)$$

In equation (60) we have the energy levels from the solution of the Dirac's equation [4]. The first term is the solution of the Schrödinger equation that we have seen above in equation (50) and the second term is the relativistic correction

$$E_{nj} = -\frac{\mu Z^2 e^4}{(4\pi\epsilon_0)^2 2\hbar^2 n^2}\frac{Z^2\alpha^2}{n}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right) \quad (61)$$

$$E_{nj} = -\frac{\mu k^4}{2\hbar^4 n^3 c^2}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right) \quad (62)$$

$$C_{nj} = \frac{\mu}{2\hbar^4 n^3 c^2}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right) \quad (63)$$

$$E_{nj} = -C_{nj}k^4 \quad (64)$$

$$E_{nj} = -C_{nj}\left(k_e + k_g\right)^4 \quad (65)$$

$$E_{nj} = -C_{nj}\left(k_e^4 + 4k_e^3k_g + 6k_e^2k_g^2 + 4k_ek_g^3 + k_g^4\right) \quad (66)$$

$$E_{nj} = -C_{nj}\left(k_e^4 + 2k_e^3k_g + 3k_e^2k_g^2 + 2k_ek_g^3 + k_g^4 + 2k_g^3k_e + 3k_g^2k_e^2 + 2k_gk_e^3\right) \quad (67)$$

$$E_{nj}^e = -C_{nj}\left(k_e^4 + 2k_e^3k_g + 3k_e^2k_g^2 + 2k_ek_g^3\right) \quad (68)$$

$$E_{nj}^g = -C_{nj}\left(k_g^4 + 2k_g^3k_e + 3k_g^2k_e^2 + 2k_gk_e^3\right) \quad (69)$$

$$E_{nj} = E_{nj}^e + E_{nj}^g \quad (70)$$

$$h\nu_{nj}^e = \left(C_{nj}^f - C_{nj}^i\right)\left(k_e^4 + 2k_e^3k_g + 3k_e^2k_g^2 + 2k_ek_g^3\right) \quad (71)$$

$$h_g\nu_{nj}^g = \left(C_{nj}^f - C_{nj}^i\right)\left(k_g^4 + 2k_g^3k_e + 3k_g^2k_e^2 + 2k_gk_e^3\right) \quad (72)$$

From equation (71) ν_{nj}^e is the relativistic correction of the electromagnetic photon emitted from the initial energy level to the final energy level and from equation (72) ν_{nj}^g is the relativistic correction of the gravitomagnetic photon emitted from the initial energy level to the final energy level.

Multilevel operator D^{mul}

We are using Pauli matrices σ, electromagnetic four-potential A_μ and charge e with $\hbar = c = 1$

$$I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\alpha = \begin{pmatrix}0 & \sigma\\ \sigma & 0\end{pmatrix} \quad \Sigma = \begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix} \quad (73)$$

In four dimensions, Minkowski's metric $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ leads to the Clifford algebra C(1,3)[7], $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\times \mathbb{I}_{4\times4}$, Dirac matrices $\gamma^0 = \sigma_3 \otimes I, \gamma^j = i\sigma_2 \otimes \sigma_j, j = 1,2,3; \gamma^p = -i\gamma^{14} = -i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\gamma^4 = \gamma^0\gamma^1, \gamma^5 = \gamma^0\gamma^2, \gamma^6 = \gamma^0\gamma^3, \gamma^7 = \gamma^1\gamma^2, \gamma^8 = \gamma^1\gamma^3, \gamma^9 = \gamma^2\gamma^3$$

$$\gamma^{10} = \gamma^0\gamma^1\gamma^2, \gamma^{11} = \gamma^0\gamma^1\gamma^3, \gamma^{12} = \gamma^0\gamma^2\gamma^3, \gamma^{13} = \gamma^1\gamma^2\gamma^3, \gamma^{14} = \gamma^0\gamma^1\gamma^2\gamma^3 \quad (74)$$

Multilevel operator D^n acts on level n, n is the number of γ matrices in the product of the algebra members, for example, D^3 acts on $\gamma^{10},\gamma^{11},\gamma^{12}$ and γ^{13}. Total multilevel operator $D^{\mathrm{mul}} = D^0 + D^1 + D^2 + D^3 + D^4$, the action of D^{mul} on the spinor function vanishes $D^{mul}\Psi = 0$ (75)

$$D^0 = -m \quad (76)$$

$$D^1 = \gamma^\mu p_\mu - ie\gamma^\mu A_\mu \quad (77)$$

$$D^2 = -ie\gamma^\mu\gamma^\nu F_{\mu\nu} \text{ with } F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \quad (78)$$

$$D^2 = -ie\alpha E + e\Sigma H \quad (79)$$

$$D^3 = -ie\gamma^\mu\gamma^\nu\gamma^\delta F_{\mu\nu\delta} \text{ with } F_{\mu\nu\delta} = A_{\mu,\nu,\delta} - A_{\mu,\delta,\nu} = 0 \quad (80)$$

$$D^4 = -ie\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\lambda F_{\mu\nu\delta\lambda} \text{ with } F_{\mu\nu\delta\lambda} = A_{\mu,\nu,\delta,\lambda} - A_{\mu,\nu,\lambda,\delta} = 0 \quad (81)$$

Multilevel operator $D^{mul}(\eta_{\mu\nu}, A_\mu, e)$ can be generalized for a curved space with four-potential P, field charge q and covariant derivative [8]$(;\mu)$ instead of derivative $(,_\mu)$ in the definition of p_μ

$$D^{mul}(g_{\mu\nu}, P_\mu, q)\Psi = 0 \quad (82)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbb{I} \quad (83)$$

$$D^0 = -m \quad (84)$$

$$D^1 = \gamma^\mu p_\mu - iq\gamma^\mu P_\mu \quad (85)$$

$$D^2 = -iq\gamma^\mu\gamma^\nu G_{\mu\nu} \text{ with } G_{\mu\nu} = P_{\mu;\nu} - P_{\nu;\mu} \quad (86)$$

$$G_{\mu\nu}(P) = P_{\mu;\nu} - P_{\nu;\mu} = P_{\mu,\nu} - P_{\nu,\mu} + P_\alpha T^\alpha_{\mu\nu} \quad (87)$$

$$G_{\mu\nu}(P) = F_{\mu\nu}(P) + P_\alpha T^\alpha_{\mu\nu} \quad (88)$$

$$D^2 = -iq\alpha E(P) + q\Sigma H(P) - iq\frac{1}{2}\gamma^\mu\gamma^\nu P_\alpha T^\alpha_{\mu\nu} \quad (89)$$

For gravity $G_{\mu\nu}(P)$ is the new gravitomagnetic tensor. $T^\alpha_{\mu\nu}$ is the torsion tensor [9]

$$D^3 = -iq\gamma^\mu\gamma^\nu\gamma^\delta G_{\mu\nu\delta} \text{ with } G_{\mu\nu\delta} = P_{\mu;\nu;\delta} - P_{\mu;\delta;\nu} \quad (90)$$

$$G_{\mu\nu\delta}(P) = P_\alpha R^\alpha_{\mu\nu\delta} \text{ with } R^\alpha_{\mu\nu\delta} \text{ the Riemann-Christoffel tensor[10]} \quad (91)$$

$$D^3 = -iq\gamma^0\gamma^1\gamma^2 P_\alpha R^\alpha_{012} - iq\gamma^0\gamma^1\gamma^3 P_\alpha R^\alpha_{013} - iq\gamma^0\gamma^2\gamma^3 P_\alpha R^\alpha_{023} - iq\gamma^1\gamma^2\gamma^3 P_\alpha R^\alpha_{123} \quad (92)$$

$$D^4 = -iq\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\lambda G_{\mu\nu\delta\lambda} \text{ with } G_{\mu\nu\delta\lambda} = P_{\mu;\nu;\delta;\lambda} - P_{\mu;\nu;\lambda;\delta} \quad (93)$$

$$G_{\mu\nu\delta\lambda}(P) = P_{\alpha;\nu} R^\alpha_{\mu\delta\lambda} + P_{\mu;\alpha} R^\alpha_{\nu\delta\lambda} \quad (94)$$

$$D^4 = -iq\gamma^0\gamma^1\gamma^2\gamma^3 P_{\alpha;1} R^\alpha_{023} - iq\gamma^0\gamma^1\gamma^2\gamma^3 P_{0;\alpha} R^\alpha_{123} \quad (95)$$

Gravitomagnetic tensor defined in Schwarzschild's metric

We are using $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$ with $G = c = 1$, this metric is defined by [11]

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (96)$$

$$g_{00} = \left(1 - \frac{2M}{r}\right), g_{11} = -\left(1 - \frac{2M}{r}\right)^{-1}, g_{22} = -r^2, g_{33} = -r^2\sin^2\theta \quad (97)$$

$$g^{00} = \left(1 - \frac{2M}{r}\right)^{-1}, g^{11} = -\left(1 - \frac{2M}{r}\right), g^{22} = -r^{-2}, g^{33} = -r^{-2}\sin^{-2}\theta \quad (98)$$

$$\Gamma^1_{00} = g_{00} M r^{-2} \quad (99)$$

$$\Gamma^0_{01} = g_{00}^{-1} M r^{-2} \quad (100)$$

$$\Gamma^1_{11} = -g_{00}^{-1} M r^{-2} \quad (101)$$

$$\Gamma^2_{12} = \Gamma^3_{13} = r^{-1} \quad (102)$$

$$\Gamma^1_{22} = -g_{00} r \quad (103)$$

$$\Gamma^3_{23} = \cot\theta \quad (104)$$

$$\Gamma^1_{33} = -g_{00} r \sin^2\theta \quad (105)$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \quad (106)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbb{I} \quad (107)$$

$$\gamma^0 = g_{00}^{-1/2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (108)$$

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$$\gamma^1 = -g_{00}^{1/2}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \quad (109)$$

$$\gamma^2 = -r^{-1}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \quad (110)$$

$$\gamma^3 = -r^{-1}\sin^{-1}\theta\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \quad (111)$$

$D^2 = im\alpha E(P) - m\Sigma H(P) + im\frac{1}{2}\gamma^\mu\gamma^\nu P_\alpha T^\alpha_{\mu\nu}$, from equations (88) and (89) (112)

$$G_{12} = -H_3(P) = \left(-g_{00}^{1/2}\right)(-r^{-1})(P_{1,2} - P_{2,1}) \quad (113)$$

$$G_{13} = H_2(P) = \left(-g_{00}^{1/2}\right)(-r^{-1}\sin^{-1}\theta)(P_{1,3} - P_{3,1}) \quad (114)$$

$$G_{23} = -H_1(P) = (-r^{-1})(-r^{-1}\sin^{-1}\theta)(P_{2,3} - P_{3,2}) \quad (115)$$

$$G_{03} = -E_3(P) = \left(g_{00}^{-1/2}\right)(-r^{-1}\sin^{-1}\theta)(P_{0,3} - P_{3,0}) \quad (116)$$

$$G_{02} = -E_2(P) = \left(g_{00}^{-1/2}\right)(-r^{-1})(P_{0,2} - P_{2,0}) \quad (117)$$

$$G_{01} = -E_1(P) = \left(g_{00}^{-1/2}\right)\left(-g_{00}^{1/2}\right)(P_{0,1} - P_{1,0}) \quad (118)$$

$T^\alpha_{\mu\nu} = 0$ and $R^\alpha_{012} = R^\alpha_{013} = R^\alpha_{023} = R^\alpha_{123} = 0$ (119)

Energy-momentum form is a 1-form [12]

$$\mathbf{p} = E\mathbf{d}t - p_x\mathbf{d}x - p_y\mathbf{d}y - p_z\mathbf{d}z \quad (120)$$

$\mathbf{dp}$ is a 2 -form

$$\mathbf{G} = \mathbf{dp} = E_x\mathbf{d}t\wedge\mathbf{d}x + E_y\mathbf{d}t\wedge\mathbf{d}y + E_z\mathbf{d}t\wedge\mathbf{d}z - B_x\mathbf{d}y\wedge\mathbf{d}z - B_y\mathbf{d}z\wedge\mathbf{d}x - B_z\mathbf{d}x\wedge\mathbf{d}y \quad (121)$$

$$G_{32} = p_{z,y} - p_{y,z} \quad (122)$$

$$G_{13} = p_{x,z} - p_{z,x} \quad (123)$$

$$G_{21} = p_{y,x} - p_{x,y} \quad (124)$$

Comparing equations (113-115) and (122-124) we can infer $P_\alpha = p_\alpha$ (125)

D^0 is related to the scalar 0 -form m, D^1 is related to the Energy-momentum 1 -form, D^2 is related to the Electromagnetic 2 -form, D^3 is related to $^*\mathbf{J}3$-form [13]

$$\begin{pmatrix} *J_{123} \\ *J_{023} \\ *J_{013} \\ *J_{012} \end{pmatrix} = \begin{pmatrix} -\rho \\ j_1 \\ -j_2 \\ j_3 \end{pmatrix} \quad (126)$$

D^4 is related to $\mathbf{L}4$-form [14]

$$\mathbf{L} = L_{0123}\mathbf{d}x^0\wedge\mathbf{d}x^1\wedge\mathbf{d}x^2\wedge\mathbf{d}x^3 \quad (127)$$

$\gamma^p = -i\gamma^{14}$ is the proyector matrix, historically γ^5, but $\gamma^5 = \gamma^0\gamma^2$

$$\gamma^p = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \left(g_{00}^{-1/2}\right)\left(-g_{00}^{1/2}\right)(-r^{-1})(-r^{-1}\sin^{-1})\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \quad (128)$$

Torsion tensor in a rearranged Kerr's metric

We are using $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$, M is the black hole's mass and a is the angular momentum per unit mass with $G = c = 1$. The invariance of the length of vectors under parallel transport means that the connection is compatible with the metric, it is a metric connection, the requirement of the preservation of the length by parallel transport may be stated as [15]

$$g_{\mu\nu;\sigma} = 0 \quad (129)$$

$$g_{\mu\nu;\sigma} = g_{\mu\nu,\sigma} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\sigma} - g_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma} \quad (130)$$

$$0 = g_{\mu\nu,\sigma} - g_{\nu\alpha}\Gamma^{\alpha}_{\mu\sigma} - t_{\alpha\nu}\Gamma^{\alpha}_{\mu\sigma} - g_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma}, \text{ with } t_{\mu\nu} = g_{\mu\nu} - g_{\nu\mu} \quad (131)$$

$$0 = g_{\mu\nu,\sigma} - g_{\nu\alpha}\Gamma^{\alpha}_{\mu\sigma} - t_{\alpha\nu}\Gamma^{\alpha}_{\mu\sigma} - g_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma} \quad (132)$$

$$g_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma} + g_{\nu\alpha}\Gamma^{\alpha}_{\mu\sigma} + t_{\alpha\nu}\Gamma^{\alpha}_{\mu\sigma} = g_{\mu\nu,\sigma} \quad (133)$$

$$\Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma} + t_{\alpha\nu}g^{\alpha\lambda}\Gamma_{\lambda\mu\sigma} = g_{\mu\nu,\sigma} \quad (134)$$

Solving these equations we get the torsion applying its definition [16]

$$\Gamma_{\mu\nu\sigma} - \Gamma_{\mu\sigma\nu} = -T_{\mu\nu\sigma} \quad (135)$$

Expanding the line element in powers of r^{-1} and examining the leading terms [17]

$$ds^2 = \left[1 - \frac{2M}{r} + O(r^{-2})\right]dt^2 + \left[\frac{4aM}{r}\sin^2\theta + O(r^{-2})\right]dtd\phi - [1 + O(r^{-1})][dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2] \quad (136)$$

Rearranging the line elements

$$ds^2 = \left[1 - \frac{2M}{r} + O(r^{-2})\right]dt^2 + [O(r^{-2})]dtd\phi + \left[\frac{4aM}{r}\sin^2\theta\right]d\phi dt - [1 + O(r^{-1})][dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2] \quad (137)$$

$$g_{00} = \left[1 - \frac{2M}{r} + O(r^{-2})\right] \quad (138)$$

$$g_{03} = [O(r^{-2})] \quad (139)$$

$$g_{11} = -[1 + O(r^{-1})] \quad (140)$$

$$g_{22} = -[1 + O(r^{-1})]r^2 \quad (141)$$

$$g_{33} = -[1 + O(r^{-1})]r^2\sin^2\theta \quad (142)$$

$$g_{30} = \left[\frac{4aM}{r}\sin^2\theta\right] \quad (143)$$

$$t_{03} = [O(r^{-2})] - \left[\frac{4aM}{r}\sin^2\theta\right] \quad (144)$$

$$g^{00} = g_{33}(g_{00}g_{33} - g_{03}g_{30})^{-1} \quad (145)$$

$$g^{03} = -g_{03}(g_{00}g_{33} - g_{03}g_{30})^{-1} \quad (146)$$

$$g^{11} = g_{11}^{-1} \quad (147)$$

$$g^{22} = g_{22}^{-1} \quad (148)$$

$$g^{30} = -g_{30}(g_{00}g_{33} - g_{03}g_{30})^{-1} \quad (149)$$

$$g^{33} = g_{00}(g_{00}g_{33} - g_{03}g_{30})^{-1} \quad (150)$$

Generalizing Clifford algebra with $\delta^{clif}_{\mu\nu}$,if $\mu \neq \nu$ and $g_{\mu\nu} \neq 0$ then 1 ,else 0

$$\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} + g^{\nu\mu} - g^{\mu\nu}\delta^{clif}_{\mu\nu} - g^{\nu\mu}\delta^{clif}_{\nu\mu} \quad (151)$$

$$\gamma^0 = [1 + O(r^{-1})]^{1/2} r\sin\theta (g_{00}g_{33} - g_{03}g_{30})^{-1/2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (152)$$

$$\gamma^1 = [1 + O(r^{-1})]^{-1/2}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \quad (153)$$

$$\gamma^2 = [1 + O(r^{-1})]^{-1/2}r^{-1}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \quad (154)$$

$$\gamma^3 = \left[1 - \frac{2M}{r} + O(r^{-2})\right]^{1/2}(g_{00}g_{33} - g_{03}g_{30})^{-1/2}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \quad (155)$$

$$D^2 = im\alpha E(P) - m\Sigma H(P) + im\frac{1}{2}\gamma^\mu\gamma^\nu P_\alpha T^{\alpha}_{\mu\nu}, \text{ from equations (88) and (89)} \quad (156)$$

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$$F_{12} = -H_3(P) = \left([1+O(r^{-1})]^{-1/2}\right)\left([1+O(r^{-1})]^{-1/2}r^{-1}\right)\left(P_{1,2}-P_{2,1}\right) \quad (157)$$

$$F_{13} = H_2(P) = \left([1+O(r^{-1})]^{-1/2}\right)\left(\left[1-\frac{2M}{r}+O(r^{-2})\right]^{1/2}(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\right)\left(P_{1,3}-P_{3,1}\right) \quad (158)$$

$$F_{23} = -H_1(P) = \left([1+O(r^{-1})]^{-1/2}r^{-1}\right)\left(\left[1-\frac{2M}{r}+O(r^{-2})\right]^{1/2}(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\right)\left(P_{2,3}-P_{3,2}\right) \quad (159)$$

$$F_{03} = -E_3(P) = \left([1+O(r^{-1})]^{1/2}r\sin\theta(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\right)\left(\left[1-\frac{2M}{r}+O(r^{-2})\right]^{1/2}(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\right)\left(P_{0,3}-P_{3,0}\right) \quad (160)$$

$$F_{02} = -E_2(P) = \left([1+O(r^{-1})]^{1/2}r\sin\theta(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\right)\left([1+O(r^{-1})]^{-1/2}r^{-1}\right)\left(P_{0,2}-P_{2,0}\right) \quad (161)$$

$$F_{01} = -E_1(P) = \left([1+O(r^{-1})]^{1/2}r\sin\theta(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\right)\left([1+O(r^{-1})]^{-1/2}\right)\left(P_{0,1}-P_{1,0}\right) \quad (162)$$

$\gamma^p = -i\gamma^{14}$ is the proyector matrix, historically γ^5, but $\gamma^5 = \gamma^0\gamma^2$

$$\gamma^p = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \left([1+O(r^{-1})]^{1/2}r\sin\theta(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\right)\left([1+O(r^{-1})]^{-1/2}\right)\left([1+O(r^{-1})]^{-1/2}r^{-1}\right)\left(\left[1-\frac{2M}{r}+O(r^{-2})\right]^{1/2}(g_{00}g_{33}-g_{03}g_{30})^{-1/2}\right)\begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix} \quad (163)$$

Geodesic equation and torsion tensor

A geodesic that is not a null geodesic has the property that $\int ds$, taken along a section of the track with the end points P and Q, is stationary if one makes a small variation of the track keeping the end points fixed. If dx^μ denotes an element along the track [18]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad (164)$$

$$2ds\delta(ds) = dx^\mu dx^\nu\delta(g_{\mu\nu}) + g_{\mu\nu}\delta(dx^\mu)dx^\nu + g_{\mu\nu}dx^\mu\delta(dx^\nu) \quad (165)$$

$$2ds\delta(ds) = dx^\mu dx^\nu g_{\mu\nu,\lambda}\delta(x^\lambda) + 2g_{\mu\lambda}dx^\mu\delta(dx^\lambda) \quad (166)$$

$\delta(dx^\lambda) = d\delta(x^\lambda)$ and $dx^\mu = v^\mu ds$ (167)

$$\int\delta(ds) = \int\left[\frac{1}{2}g_{\mu\nu,\lambda}v^\mu v^\nu\delta x^\lambda + g_{\mu\lambda}v^\mu\frac{d\delta x^\lambda}{ds}\right]ds \quad (168)$$

By partial integration with $\delta x^\lambda = 0$ at end points P and Q, we get

$$\delta\int ds = \int\left[\frac{1}{2}g_{\mu\nu,\lambda}v^\mu v^\nu - \frac{d}{ds}(g_{\mu\lambda}v^\mu)\right]\delta x^\lambda ds \quad (169)$$

The condition for this to vanish with arbitrary δx^λ is

$$\frac{d}{ds}(g_{\mu\lambda}v^\mu) - \frac{1}{2}g_{\mu\nu,\lambda}v^\mu v^\nu = 0 \quad (170)$$

$$\frac{d}{ds}(g_{\mu\lambda}v^\mu) = g_{\mu\lambda}\frac{dv^\mu}{ds} + g_{\mu\lambda,\nu}v^\mu v^\nu \quad (171)$$

$$\frac{d}{ds}(g_{\mu\lambda}v^\mu) = g_{\lambda\mu}\frac{dv^\mu}{ds} - t_{\lambda\mu}\frac{dv^\mu}{ds} + \frac{1}{2}(g_{\mu\lambda,\nu}+g_{\nu\lambda,\mu})v^\mu v^\nu, \text{ and with } t_{\mu\lambda} = g_{\mu\lambda} - g_{\lambda\mu} \quad (172)$$

$$\frac{d}{ds}(g_{\mu\lambda}v^\mu) = g_{\lambda\mu}\frac{dv^\mu}{ds} - t_{\lambda\mu}\frac{dv^\mu}{ds} + \frac{1}{2}(g_{\lambda\mu,\nu}+g_{\lambda\nu,\mu})v^\mu v^\nu + \frac{1}{2}(t_{\mu\lambda,\nu}+t_{\nu\lambda,\mu})v^\mu v^\nu \quad (173)$$

From equation (170) with $t_{\mu\nu,\lambda} = g_{\mu\nu,\lambda} - g_{\nu\mu,\lambda}$

$$\frac{d}{ds}(g_{\mu\lambda}v^\mu) - \frac{1}{2}g_{\nu\mu,\lambda}v^\mu v^\nu - \frac{1}{2}t_{\mu\nu,\lambda}v^\mu v^\nu = 0 \quad (174)$$

Thus the condition (174) becomes

$$g_{\lambda\mu}\frac{dv^\mu}{ds} - t_{\lambda\mu}\frac{dv^\mu}{ds} + \bar{\Gamma}_{\lambda\mu\nu}v^\mu v^\nu - \dot{\Gamma}_{\lambda\mu\nu}v^\mu v^\nu = 0 \quad (175)$$

$$\bar{\Gamma}_{\lambda\mu\nu} = \frac{1}{2}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}) \quad (176)$$

$$\dot{\Gamma}_{\lambda\mu\nu} = \frac{1}{2}(t_{\lambda\mu,\nu} + t_{\lambda\nu,\mu} - t_{\nu\mu,\lambda}) \quad (177)$$

Multiplying equation (175) by $g^{\sigma\lambda}$, we obtain the geodesic equation

$$\frac{dv^\sigma}{ds} - g^{\sigma\lambda}t_{\lambda\mu}\frac{dv^\mu}{ds} + \bar{\Gamma}^\sigma_{\mu\nu}v^\mu v^\nu - \dot{\Gamma}^\sigma_{\mu\nu}v^\mu v^\nu = 0 \quad (178)$$

$$\frac{dv^\sigma}{ds} - g^{\sigma\lambda}t_{\lambda\mu}\frac{dv^\mu}{ds} + \Gamma^\sigma_{\mu\nu}v^\mu v^\nu = 0 \quad (179)$$

$$\Gamma^\sigma_{\mu\nu} = \bar{\Gamma}^\sigma_{\mu\nu} - \dot{\Gamma}^\sigma_{\mu\nu} \quad (180)$$

$\bar{\Gamma}^\sigma_{\mu\nu}$ are the Christoffel symbols of the symmetric part, so

$$-\bar{T}^\sigma_{\mu\nu} = \bar{\Gamma}^\sigma_{\mu\nu} - \bar{\Gamma}^\sigma_{\nu\mu} = \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu} + \dot{\Gamma}^\sigma_{\mu\nu} - \dot{\Gamma}^\sigma_{\nu\mu} \quad (181)$$

$$0 = \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu} + \dot{\Gamma}^\sigma_{\mu\nu} - \dot{\Gamma}^\sigma_{\nu\mu} \quad (182)$$

We directly obtain the torsion tensor without solving equations (134) and (135)

$$T^\sigma_{\mu\nu} = \dot{\Gamma}^\sigma_{\mu\nu} - \dot{\Gamma}^\sigma_{\nu\mu} \quad (183)$$

Einstein field equation and conservation laws

From equation (180) where $\bar{\Gamma}^\sigma_{\mu\nu}$ are the symbols of the symmetric part

$$\Gamma^\mu_{\nu\mu} = \bar{\Gamma}^\mu_{\nu\mu} - \dot{\Gamma}^\mu_{\nu\mu} \quad (184)$$

$$\bar{\Gamma}^\mu_{\nu\mu} = \bar{\Gamma}^\mu_{\mu\nu} = g^{\mu\lambda}\bar{\Gamma}_{\lambda\mu\nu} = g^{\mu\lambda}\frac{1}{2}\left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}\right) \quad (185)$$

$$\bar{\Gamma}^\mu_{\nu\mu} = (\sqrt{-g})^{-1}\sqrt{-g}_{,\nu} + g^{\mu\lambda}\frac{1}{2}\left(g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}\right) \quad (186)$$

Equation (184) becomes

$$\Gamma^\mu_{\nu\mu} = (\sqrt{-g})^{-1}\sqrt{-g}_{,\nu} + g^{\mu\lambda}\frac{1}{2}\left(g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}\right) - \dot{\Gamma}^\mu_{\nu\mu} \quad (187)$$

$$\Gamma^\mu_{\nu\mu} = (\sqrt{-g})^{-1}\sqrt{-g}_{,\nu} - t_\nu \quad (188)$$

$$t_\nu = -g^{\mu\lambda}\frac{1}{2}\left(g_{\lambda\nu},\mu - g_{\nu\mu,\lambda}\right) + \dot{\Gamma}^\mu_{\nu\mu} \quad (189)$$

The vector A^μ has the covariant divergence

$$A^\mu_{;\mu} = A^\mu_{,\mu} + \Gamma^\mu_{\nu\mu}A^\nu \quad (190)$$

$$\left(A^\mu_{;\mu} + t_\nu A^\nu\right)\sqrt{-g} = \left(A^\mu\sqrt{-g}\right)_{,\mu} \quad (191)$$

If the left-hand side of equation (191) equals zero then the right-hand side gives us the first conservation law.

For the antisymmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$

$$F^{\mu\nu}_{;\nu} = F^{\mu\nu}{}_{,\nu} + \Gamma^\mu_{\rho\nu}F^{\rho\nu} + \Gamma^\nu_{\rho\nu}F^{\mu\rho} \quad (192)$$

$$F^{\mu\nu}_{;\nu} = F^{\mu\nu}{}_{,\nu} - T^\mu_{\rho\nu}F^{\rho\nu} + \left((\sqrt{-g})^{-1}\sqrt{-g}_{,\rho} - t_\rho\right)F^{\mu\rho} \quad (193)$$

$$\left(F^{\mu\nu}_{;\nu} + T^\mu_{\rho\nu}F^{\rho\nu} + t_\rho F^{\mu\rho}\right)\sqrt{-g} = \left(F^{\mu\nu}\sqrt{-g}\right)_{,\nu} \quad (194)$$

If the left-hand side of equation (194) equals zero then the right-hand side gives us the second conservation law.

For the antisymmetric tensor $F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu}$

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} + T^\alpha_{\mu\nu}A_\alpha \quad (195)$$

$$F_{\mu\nu;\sigma} = F_{\mu\nu,\sigma} - \Gamma^\alpha_{\mu\sigma}F_{\alpha\nu} - \Gamma^\alpha_{\nu\sigma}F_{\mu\alpha} \quad (196)$$

$$F_{\nu\sigma;\mu} = F_{\nu\sigma,\mu} - \Gamma^\alpha_{\nu\mu}F_{\alpha\sigma} - \Gamma^\alpha_{\sigma\mu}F_{\nu\alpha} \quad (197)$$

$$F_{\sigma\mu;\nu} = F_{\sigma\mu,\nu} - \Gamma^\alpha_{\sigma\nu}F_{\alpha\mu} - \Gamma^\alpha_{\mu\nu}F_{\sigma\alpha} \quad (198)$$

Adding equations (196), (197) and (198)

$$F_{\mu\nu;\sigma} + F_{\nu\sigma;\mu} + F_{\sigma\mu;\nu} = T^{\alpha}_{\mu\nu}F_{\sigma\alpha} + \left(T^{\alpha}_{\mu\nu}A_{\alpha}\right)_{,\sigma} + T^{\alpha}_{\nu\sigma}F_{\mu\alpha} + (T^{\alpha}_{\nu\sigma}A_{\alpha})_{,\mu} + T^{\alpha}_{\sigma\mu}F_{\nu\alpha} + \left(T^{\alpha}_{\sigma\mu}A_{\alpha}\right)_{,\nu} \quad (199)$$

From the definition of the curvature tensor $R^{\beta}_{\nu\rho\sigma}$

$$R^{\beta}_{\nu\rho\sigma} = \Gamma^{\beta}_{\nu\sigma,\rho} - \Gamma^{\beta}_{\nu\rho,\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\beta}_{\alpha\sigma} \quad (200)$$

$R^{\mu}_{\nu\mu\rho}$ is called the Ricci tensor

$$R_{\mu\nu} = -\Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} \quad (201)$$

Now $R_{\mu\nu}$ is not symmetric, $2A_{\mu\nu} = R_{\mu\nu} - R_{\nu\mu}$ is the antisymmetric part and $R_{\mu\nu} = A_{\mu\nu} + S_{\mu\nu}$ where $S_{\mu\nu}$ is the symmetric part in the Einstein's equation [19].

$$S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S = \kappa T_{\mu\nu} \quad (202)$$

$$2A_{\mu\nu} = -\left((\sqrt{-g})^{-1}\sqrt{-g}_{,\mu} - t_{\mu}\right)_{,\nu} + \left((\sqrt{-g})^{-1}\sqrt{-g}_{,\nu} - t_{\nu}\right)_{,\mu} - T^{\alpha}_{\mu\nu,\alpha} \quad - T^{\alpha}_{\mu\nu}\left((\sqrt{-g})^{-1}\sqrt{-g}_{,\alpha} - t_{\alpha}\right) \quad (203)$$

$$2A_{\mu\nu} = t_{\mu,\nu} - t_{\nu,\mu} - T^{\alpha}_{\mu\nu,\alpha} - T^{\alpha}_{\mu\nu}\left((\sqrt{-g})^{-1}\sqrt{-g}_{,\alpha} - t_{\alpha}\right) \quad (204)$$

$$R_{\mu\nu} - A_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - A) = \kappa T_{\mu\nu} \quad (205)$$

$$R_{\mu\nu} - A_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \quad (206)$$

Conclusions

Multilevel operator $D^{mul}\left(g_{\mu\nu}, P_{\mu}, q\right)$ has been generalized for a curved space with a general four-potential P. For gravity $G_{\mu\nu}(P)$ is the new gravitomagnetic tensor and torsion tensor $T^{\alpha}_{\mu\nu}$ appears in its definition.

In a flat space $G_{\mu\nu}(A) = F_{\mu\nu}(A), D^3$ and D^4 operators vanish. In a curved space the curvature tensor $R^{\alpha}_{\mu\nu\delta}$ appears in levels 3 and 4.

The appearance of torsion tensor $T^{\alpha}_{\mu\nu}$ and curvature tensor $R^{\alpha}_{\mu\nu\delta}$ in multilevel operator $D^{mul}\left(g_{\mu\nu}, P_{\mu}, q\right)$ means that this operator is a fundamental operator in Quantum Field Theory.

$\gamma^0, \gamma^1, \gamma^2, \gamma^3$, have been calculated for Schwarzschild's metric, then $G_{\mu\nu}(P)$, the gravitomagnetic tensor has been obtained.

Each D^n, where n is the number of γ matrices in the product of the algebra members, is related to an n-form

The invariance of the length of vectors under parallel transport requires the vanishing of the metric tensor covariant derivative, a new term appears $t_{\mu\nu} = g_{\mu\nu} - g_{\nu\mu}$ measuring the non symmetric part of the metric tensor, solving these equations we get the torsion tensor.

Rearranging Kerr's metric we obtained $t_{03} = g_{03} - g_{30}$, the non symmetric part of the metric tensor, gravitomagnetic tensor has also been calculated generalizing the Clifford algebra.

Taking into account the $t_{\mu\nu} = g_{\mu\nu} - g_{\nu\mu}$ in the geodesic equation we have obtained the torsion tensor, conservation laws and Einstein field equation in a non-symmetric geometry.

The solution of Schrödinger's equation leads us to the emission of gravitomagnetic photons when the gravitational potential is also taken into account. The correction of the second term in electromagnetic photon frequency is an indirect detection of the gravitomagnetic photon emission.

The solution of Schrödinger's equation for a gravitational system leads us to Planck's gravitational constant value and Kepler's third law.

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